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Boundary value problems for a third order equation with multiple characteristics in three-dimensional space in semi-bounded domains

Yu. P. Apakov[a,*], A. A. Hamitov[b]

[a]*Romanovskii Institute of Mathematics, Academy of Sciences of the Republic of Uzbekistan, Tashkent, 100174 Uzbekistan*
[b]*Namangan Engineering-Construction Institute, Namangan, 160103 Uzbekistan*

Abstract. In this article, for a third-order equation with multiple characteristics, boundary value problems in three-dimensional space in semi-bounded domains are formulated and studied. The uniqueness of the solution is proven by the method of energy integrals. The existences of solutions is proved by the method of separation of variables. The solutions are constructed explicitly in the form of an infinite series, and the possibility of term-by-term differentiation of the series with respect to all variables is justified.

1. Introduction

Third-order partial differential equations are considered when solving problems in the theory of non-linear acoustics and in the hydrodynamic theory of space plasma and fluid filtration in porous media [1],[2].

In the work [3], taking into account the properties of viscosity and thermal conductivity of the gas, a third-order equation with multiple characteristics was obtained from the Navier-Stokes system, containing the second derivative with respect to time

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \ \ \nu = const.$$

This equation when $\nu = 1$ describes an axisymmetric flow, and when $\nu = 0$ it describes a plane-parallel flow [4].

The first results on a third-order equation with multiple characteristics were obtained in the works of H. Block [5], E. Del Vecehio [6].

L. Catabriga in the work [7] for equation $D_x^{2n+1}u - D_y^2 u = 0$ constructed a fundamental solution in the form of a double improper integral and studied the properties of the potential and solved boundary value problems.

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* Corresponding author: Yu. P. Apakov
Email addresses: yusupjonapakov@gmail.com (Yu. P. Apakov), azizbek.khamitov.93@mail.ru (A. A. Hamitov)
ORCID iDs: https://orcid.org/0000-0001-8805-8917 (Yu. P. Apakov), https://orcid.org/0009-0000-2184-450X (A. A. Hamitov)

In the works [8],[9], fundamental solutions of a third-order equation with multiple characteristics were constructed, containing second derivatives with respect to time, expressed through degenerate hypergeometric functions, their properties were studied, and estimates were found for $|t| \to \infty$.

In works [10], [11], [12], [13], [14], [15], [16], boundary value problems for third-order equations were studied.

2. Statement of the problem

In the domains $D^+ = \{(x, y, z) : 0 < x < +\infty,\ 0 < y < q,\ 0 < z < r\}$ and $D^- = \{(x, y, z) : -\infty < x < 0,\ 0 < y < q,\ 0 < z < r\}$, consider the equation

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0, \tag{1}$$

where $q > 0,\ r > 0$ - constant real numbers, and for it, we will study the following problems.

Problem 2.1. *Find a solution to equation (1) in the domain D^+ from class $C^{3,2,2}_{x,y,z}(D^+) \cap C^{2,1,1}_{x,y,z}(D^+ \cup \Gamma_1)$, having a limited second derivative with respect by x, as $x \to +\infty$ and $u_y, u_z \in L_2(D^+)$, satisfying the following boundary conditions:*

$$\begin{cases} \alpha u(x,0,z) + \beta u_y(x,0,z) = 0, \\ \gamma u(x,q,z) + \delta u_y(x,q,z) = 0, \quad 0 < x < +\infty, \\ u(x,y,0) = u(x,y,r) = 0, \end{cases} \tag{2}$$

$$u(0,y,z) = \psi_1(y,z), \lim_{x\to+\infty} u(x,y,z) = \lim_{x\to+\infty} u_x(x,y,z) = 0, uniformly\ limited\ by\ 0 \le y \le q,\ 0 \le z \le r, \tag{3}$$

where $\Gamma_1 = \partial D^+$ - boundary of the domain D^+, $\alpha,\ \beta,\ \gamma,\ \delta \in R\backslash\{0\}$, and $\psi_1(y,z)$ - a given sufficiently smooth function, moreover

$$\begin{cases} \alpha \dfrac{\partial^j \psi_1(0,z)}{\partial y^j} + \beta \dfrac{\partial^{j+1} \psi_1(0,z)}{\partial y^{j+1}} = 0,\ \gamma \dfrac{\partial^j \psi_1(q,z)}{\partial y^j} + \delta \dfrac{\partial^{j+1} \psi_1(q,z)}{\partial y^{j+1}} = 0, \\ \dfrac{\partial^4 \psi_1(y,r)}{\partial y^4} = \dfrac{\partial^4 \psi_1(y,0)}{\partial y^4} = 0, \quad \dfrac{\partial^6 \psi_1(y,r)}{\partial y^4 \partial z^2} = \dfrac{\partial^6 \psi_1(y,0)}{\partial y^4 \partial z^2} = 0, \end{cases} \quad j = 0, 2. \tag{4}$$

Problem 2.2. *Find a solution to equation (1) in the domain D^- from class $C^{3,2,2}_{x,y,z}(D^-) \cap C^{2,1,1}_{x,y,z}(D^- \cup \Gamma_2)$, having bounded first and second derivatives with respect by x, as $x \to -\infty$ and $u_y, u_z \in L_2(D^-)$, satisfying the boundary conditions for $-\infty < x < 0$ (2) and*

$$u(0,y,z) = \psi_2(y,z), u_x(0,y,z) = \psi_3(y,z), \lim_{x\to-\infty} u(x,y,z) = 0,\ uniformly\ limited\ by\ 0 \le y \le q,\ 0 \le z \le r, \tag{5}$$

where $\Gamma_2 = \partial D^-$ - boundary of the domain D^-, $\alpha,\ \beta,\ \gamma,\ \delta \in R\backslash\{0\}$ and $\psi_i(y,z),\ i = 2, 3$ - given sufficiently smooth functions, moreover

$$\begin{cases} \alpha \dfrac{\partial^j \psi_i(0,z)}{\partial y^j} + \beta \dfrac{\partial^{j+1} \psi_i(0,z)}{\partial y^{j+1}} = 0, \quad \gamma \dfrac{\partial^j \psi_i(q,z)}{\partial y^j} + \delta \dfrac{\partial^{j+1} \psi_i(q,z)}{\partial y^{j+1}} = 0, \quad j = 0, 2, \\ \dfrac{\partial^4 \psi_i(y,0)}{\partial y^4} = \dfrac{\partial^4 \psi_i(y,r)}{\partial y^4} = 0, \qquad \dfrac{\partial^6 \psi_i(y,0)}{\partial y^4 \partial z^2} = \dfrac{\partial^6 \psi_i(y,r)}{\partial y^4 \partial z^2} = 0, \quad i = 2, 3. \end{cases} \tag{6}$$

We note that semi-bounded domains in the plane were studied in [17], [18], [19], [20], and in three-dimensional space for a second-order equation in [21], [22] some well-posed boundary value problems were studied. And also in works [23], [24] in finite domains, other boundary value problems in three-dimensional space were studied, and in works [25], [26] boundary value problems for a fourth-order equation in three-dimensional space were considered.

3. Uniqueness of solution

Theorem 3.1. *If the Problem 2.1 and Problem 2.2 have solutions, then if conditions $\alpha\beta < 0$, $\gamma\delta > 0$ are met, they are unique.*

Proof. Let us assume conversely, that is, let Problems 2.1 (2.2) has two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$. Then, the function $u(x,y,z) = u_1(x,y,z) - u_2(x,y,z)$ satisfies equation (1) with homogeneous boundary conditions. Then, let us prove $u(x,y,z) \equiv 0$ is D^+ (D^-).

For this purpose, we multiply both sides of equation (1) by u, and then, we get

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) - \frac{\partial}{\partial y}\left(u\,u_y\right) + u_y^2 - \frac{\partial}{\partial z}\left(u\,u_z\right) + u_z^2 = 0. \tag{7}$$

Integrate equality (7) in the domain $D_d = \{(x,y,z): \quad 0 < x < d, \quad 0 < y < q, \quad 0 < z < r\}$, where $d > 0$, and we have

$$\begin{aligned}
&\int_0^q\int_0^r u(d,y,z)\,u_{xx}(d,y,z)\,dydz - \int_0^q\int_0^r u(0,y,z)\,u_{xx}(0,y,z)\,dydz - \\
&-\frac{1}{2}\int_0^q\int_0^r u_x^2(d,y,z)\,dydz + \frac{1}{2}\int_0^q\int_0^r u_x^2(0,y,z)\,dydz - \int_0^d\int_0^r u(x,q,z)\,u_y(x,q,z)\,dxdz + \\
&+\int_0^d\int_0^r u(x,0,z)\,u_y(x,0,z)\,dxdz - \int_0^d\int_0^q u(x,y,r)\,u_z(x,y,r)\,dxdy + \\
&+\int_0^d\int_0^q u(x,y,0)\,u_z(x,y,0)\,dxdy + \iiint_{D_d} u_y^2(x,y,z)\,dxdydz + \\
&+\iiint_{D_d} u_z^2(x,y,z)\,dxdydz = 0.
\end{aligned} \tag{8}$$

If $d \to +\infty$, then $D_d \to D^+$. Moreover, taking into account the homogeneous boundary conditions of Problem 2.1, for $x \to +\infty$ and $u_y, u_z \in L_2(D^+)$, from (8) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^q\int_0^r u_x^2(0,y,z)dydz - \frac{\beta}{\alpha}\int_0^{+\infty}\int_0^r u_y^2(x,0,z)dxdz + \frac{\delta}{\gamma}\int_0^{+\infty}\int_0^r u_y^2(x,q,z)dxdz + \\
&+\iiint_{D^+} u_y^2(x,y,z)\,dxdydz + \iiint_{D^+} u_z^2(x,y,z)\,dxdydz = 0.
\end{aligned}$$

Taking into account conditions $\alpha\beta < 0$, $\gamma\delta > 0$, we obtain $u_y(x,y,z) = 0$ and $u_z(x,y,z) = 0$, i.e. $u(x,y,z) = f(x)$ in D^+. Putting into equation (1), we have $f'''(x) = 0$. Hence, $f(x) = C_1x^2 + C_2x + C_3$. From the conditions (3), we get $f(0) = 0$, $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} f'(x) = 0$, then, $C_1 = 0$, $C_2 = 0$ and $C_3 = 0$, hence we have that $f(x) = 0$. Therefore, $u(x,y,z) \equiv 0$ in $D^+ \cup \Gamma_1$. As this result, and therefore, we attain $u_1(x,y,z) = u_2(x,y,z)$.

Now, integrating the equality (7) in the domain $D_c = \{(x,y,z): \ c < x < 0, \ 0 < y < q, \ 0 < z < r\}$, where $c < 0$, we have

$$\begin{aligned}
&\int_0^q\int_0^r u(0,y,z)\,u_{xx}(0,y,z)\,dydz - \int_0^q\int_0^r u(c,y,z)\,u_{xx}(c,y,z)\,dydz - \\
&-\frac{1}{2}\int_0^q\int_0^r u_x^2(0,y,z)\,dydz + \frac{1}{2}\int_0^q\int_0^r u_x^2(c,y,z)\,dydz - \int_c^0\int_0^r u(x,q,z)\,u_y(x,q,z)\,dxdz + \\
&+\int_c^0\int_0^r u(x,0,z)\,u_y(x,0,z)\,dxdz - \int_c^0\int_0^q u(x,y,r)\,u_z(x,y,r)\,dxdy + \\
&+\int_c^0\int_0^q u(x,y,0)\,u_z(x,y,0)\,dxdy + \iiint_{D_c} u_y^2(x,y,z)\,dxdydz + \\
&+\iiint_{D_c} u_z^2(x,y,z)\,dxdydz = 0.
\end{aligned}$$

If $c \to -\infty$, then $D_c \to D^-$. Moreover, taking into account the homogeneous boundary conditions of Problem 2.2, for $x \to -\infty$ and $u_y, u_z \in L_2(D^-)$, we obtain

$$\frac{1}{2}\lim_{c\to-\infty}\int_0^q\int_0^r u_x^2(c,y,z)dydz - \frac{\beta}{\alpha}\int_{-\infty}^0\int_0^r u_y^2(x,0,z)dxdz + \frac{\delta}{\gamma}\int_{-\infty}^0\int_0^r u_y^2(x,q,z)dxdz + \\ + \iiint_{D^-} u_y^2(x,y,z)\,dxdydz + \iiint_{D^-} u_z^2(x,y,z)\,dxdydz = 0.$$

As above, it easily follows from here that $u(x,y,z) \equiv 0$ in $D^- \cup \Gamma_2$.
The proof has been completed. □

4. Existence of the solution

Theorem 4.1. *If* $\dfrac{\partial^7\psi_i(y,z)}{\partial y^4\partial z^3} \in C[0<y<q, 0<z<r]$, $i = 1,2,3$, *and these functions satisfy the conditions (4) and (6) then the solutions to the Problem 2.1 and Problem 2.2 exists.*

Proof. In order to prove the existence of a solution to the Problem 2.1 (Problem 2.2), we look for it in the form

$$u(x,y,z) = X(x)\,V(y,z). \tag{9}$$

Putting (9) into equation (1) and separating the variables, with respect to function $X(x)$ we obtain the equation:

$$X''' + \lambda X = 0, \tag{10}$$

and for function $V(y,z)$ - the following boundary value problem:

$$\begin{cases} V_{yy} + V_{zz} + \lambda V = 0, \\ \alpha V(0,z) + \beta V_y(0,z) = 0, \\ \gamma V(q,z) + \delta V_y(q,z) = 0, \\ V(y,0) = V(y,r) = 0, \end{cases} \tag{11}$$

where λ is the separation parameter. Let us find the eigenvalues and eigenfunctions of problem (11). Let's put

$$V(y,z) = Y(y)\,Z(z). \tag{12}$$

Substituting (12) into equation (11), separating the variables, we have the problems

$$\begin{cases} Y'' + \nu Y = 0, \\ \alpha Y(0) + \beta Y'(0) = 0, \\ \gamma Y(q) + \delta Y'(q) = 0, \end{cases} \tag{13}$$

$$\begin{cases} Z'' + \mu Z = 0, \\ Z(0) = Z(r) = 0, \end{cases} \tag{14}$$

where $\nu > 0$ and $\mu > 0$ are constants related by $\lambda = \nu + \mu$.

It is known from [20] that the eigenvalues of problem (13) exist only for $\nu_n > 0$ and $\nu_n = O\left(n^2\right)$, and the corresponding eigenfunctions have the form

$$Y_n(y) = \left(\alpha \sin\sqrt{\nu_n}y - \beta\sqrt{\nu_n}\cos\sqrt{\nu_n}y\right)A_n,$$

where A_n are arbitrary constants, and for problem (14) we obtain

$$Z_m(z) = A_m \sin \sqrt{\mu_m} z,$$

where A_m are arbitrary constants and $\mu_m = \left(\frac{m\pi}{r}\right)^2$.

Taking into account (12), we define

$$V_{n,m}(y,z) = A_{n,m} Y_n(y) \sin \frac{m\pi z}{r}, \tag{15}$$

where $A_{n,m}$ are some constant factors. We choose them as in [20], so that the norm of function $V_{n,m}(y,z)$ is equal to one. Orthogonality $\{V_{n,m}\}$ is easy to check

$$\int_0^q \int_0^r V_{n,m}^2(y,z)\,dydz = A_{n,m}^2 \int_0^q \left(\alpha \sin \sqrt{\nu_n} y - \beta \sqrt{\nu_n} \cos \sqrt{\nu_n} y\right)^2 dy \int_0^r \sin^2\left(\frac{m\pi}{r} z\right) dz = 1,$$

$$\begin{aligned} \left\|V_{n,m}\right\|^2 = \int_0^q \left(\alpha \sin \sqrt{\nu_n} y - \beta \sqrt{\nu_n} \cos \sqrt{\nu_n} y\right)^2 dy \int_0^r \sin^2\left(\frac{m\pi}{r} z\right) dz = \\ = \left[\frac{1}{2}\left(\alpha^2 q + \beta^2 q \nu_n - \alpha\beta\right) + \left(\frac{\beta^2 \sqrt{\nu_n}}{4} - \frac{\alpha^2}{4\sqrt{\nu_n}}\right) \sin 2\sqrt{\nu_n} q + \right. \\ \left. + \frac{\alpha\beta}{2} \cos 2\sqrt{\nu_n} q\right] \frac{r}{2}. \end{aligned}$$

Then, as a solution to the spectral problem (13), (14), we take the functions

$$V_{n,m}(y,z) = \frac{1}{\left\|V_{n,m}\right\|^2} Y_n(y) \sin \frac{m\pi z}{r}, \tag{16}$$

which correspond to the eigenvalues

$$\lambda_{n,m} = \nu_n + \left(\frac{\pi m}{r}\right)^2, n, m \in N.$$

The possibilities of expanding given functions according to (16) were proven in [20], [27]. The general solution to equation (10) has the form:

$$X_{n,m}(x) = C_{1n,m} e^{-k_{n,m} x} + e^{\frac{1}{2} k_{n,m} x} \left(C_{2n,m} \cos \frac{\sqrt{3}}{2} k_{n,m} x + C_{3n,m} \sin \frac{\sqrt{3}}{2} k_{n,m} x\right), \tag{17}$$

where

$$k_{n,m} = \sqrt[3]{\lambda_{n,m}} = \sqrt[3]{\nu_n + \mu_m}.$$

Further, from the statement of the Problem 2.1 it follows that

$$\lim_{x \to +\infty} X_{n,m}(x) = \lim_{x \to +\infty} X'_{n,m}(x) = 0.$$

Therefore, in (17) it is necessary to consider that $C_{2n,m} = C_{3n,m} = 0$. Then, the function (17) takes the form

$$X_{n,m}(x) = C_{1n,m} e^{-k_{n,m} x}. \tag{18}$$

Now, by virtue of (9), we search for the solution to the Problem 2.1 in the form

$$u(x,y,z) = \sum_{n,m=1}^{+\infty} C_{1n,m} e^{-k_{n,m} x} V_{n,m}(y,z). \tag{19}$$

The function defined by the formal series (19) satisfies conditions (2).

Assuming temporarily that the series in (19) and its derivatives converge uniformly and requiring the function $u(x,y,z)$ defined by series (19), to satisfy the boundary conditions (3), we obtain

$$u(0,y,z) = \psi_1(y,z) = \sum_{m,n=1}^{+\infty} C_{1n,m} V_{n,m}(y,z),$$

where $C_{1n,m}$ - the Fourier coefficients of the function $\psi_1(y,z)$, that is

$$C_{1n,m} = \psi_{1n,m} = \frac{1}{\left\|V_{n,m}\right\|^2} \int_0^q \int_0^r \psi_1(y,z)\, Y_n(y) \sin\frac{m\pi z}{r} dydz. \tag{20}$$

Substituting $C_{1n,m}$ into (19), we get

$$u(x,y,z) = \sum_{n,m=1}^{+\infty} \psi_{1n,m} e^{-k_{n,m}x} V_{n,m}(y,z). \tag{21}$$

Now, we prove that the series (21) and its derivatives u_{xxx}, u_{yy} and u_{zz} converge uniformly in the domain $D^+ \cup \Gamma_1$, then function $u(x,y,z)$, defined by the series (21), gives a solution to the Problem 2.1.

Let us prove the absolute and uniform convergence of the series (21). From (21) we have

$$\left|u(x,y,z)\right| \le M \sum_{n,m=1}^{+\infty} \left|\psi_{1n,m} Y_n(y)\right|. \tag{22}$$

In what follows the maximum value of all found positive known numbers in estimates will be denoted by M.

We estimate the expression $\left|\psi_{1n,m} Y_n(y)\right|$:

$$\left|\psi_{1n,m} Y_n(y)\right| \le \left|\psi_{1n,m}\right| \left|Y_n(y)\right| = \left|Y_n(y)\right| \frac{1}{\left\|V_{n,m}\right\|^2} \int_0^q \int_0^r \left|\psi_1(y,z)\right| \left|Y_n(y)\right| dydz,$$

$$\left|Y_n(y)\right| = \left|\alpha \sin\sqrt{\nu_n} y - \beta\sqrt{\nu_n}\cos\sqrt{\nu_n} y\right| \le \sqrt{\alpha^2 + \beta^2 \nu_n}.$$

Then, we have

$$\left|\psi_{1n,m} Y_n(y)\right| \le \frac{\alpha^2 + \beta^2\nu_n}{\left\|V_{n,m}\right\|^2} \int_0^q \int_0^r \left|\psi_1(y,z)\right| dydz.$$

Let us prove that the expression $\dfrac{\alpha^2 + \beta^2\nu_n}{\left\|V_{n,m}\right\|^2}$ as $n,m \to \infty$ (that is as $\nu_n \to \infty$) is bounded:

$$\frac{\alpha^2 + \beta^2\nu_n}{\left\|V_{n,m}\right\|^2} = \frac{\alpha^2 + \beta^2\nu_n}{\left[\frac{1}{2}\left(\alpha^2 q + \beta^2 q\nu_n - \alpha\beta\right) + \left(\frac{\beta^2\sqrt{\nu_n}}{4} - \frac{\alpha^2}{4\sqrt{\nu_n}}\right)\sin 2\sqrt{\nu_n} q + \frac{\alpha\beta}{2}\cos 2\sqrt{\nu_n} q\right]\frac{r}{2}}.$$

For $n,m \to \infty$ holds

$$\lim_{n,m\to\infty} \frac{\alpha^2 + \beta^2\nu_n}{\left\|V_{n,m}\right\|^2} = \frac{4\beta^2}{\beta^2 qr} = \frac{4}{qr}.$$

Hence, we conclude that for any ν_n the inequality

$$\left|\psi_{1n,m}Y_n(y)\right| \le \frac{4}{qr}\int\limits_0^q\int\limits_0^r \left|\psi_1(y,z)\right| dydz$$

is true. Let, taking into account condition (4) and integrating by parts (20), we obtain

$$\left|\psi_{1n,m}Y_n(y)\right| \le M\frac{\left|\psi^{(7)}_{1n,m}\right|}{n^4m^3}. \tag{23}$$

Taking these estimates into account, from (22) we have

$$\left|u(x,y,z)\right| \le M\sum_{n,m=1}^{+\infty}\frac{\left|\psi^{(7)}_{1n,m}\right|}{n^4m^3} < \infty,$$

It follows that the series (21) converges absolutely and uniformly.

Now, we prove that the derivatives of the series (21) included in equation (1) also converge absolutely and uniformly in the domain $D^+ \cup \Gamma_1$. To do this, from (21), we calculate the derivatives by y and z, and we attain

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{n,m=1}^{+\infty}\nu_n\psi_{1n,m}e^{-k_{n,m}x}V_{n,m}(y,z),$$
$$\frac{\partial^2 u}{\partial z^2} = -\left(\frac{\pi}{r}\right)^2\sum_{n,m=1}^{+\infty}m^2\psi_{1n,m}e^{-k_{n,m}x}V_{n,m}(y,z).$$

Let us estimate the resulting equalities and, taking into account (23), for $x > 0$, we have

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M\sum_{n,m=1}^{+\infty}\frac{\left|\psi^{(7)}_{1n,m}\right|}{n^2m^3} < \infty,$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\sum_{n,m=1}^{+\infty}\frac{\left|\psi^{(7)}_{1n,m}\right|}{n^4m}.$$

Using the Cauchy-Bunyakovsky and Bessel inequality, we obtain

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\sqrt{\sum_{m,n=1}^{+\infty}\left|\psi^{(7)}_{1n,m}\right|^2}\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n^4m}\right)^2} \le M\left\|\frac{\partial^7\psi_1(y,z)}{\partial y^4\partial z^3}\right\| < \infty,$$

where

$$\sum_{n,m=1}^{+\infty}\left|\psi^{(7)}_{1n,m}\right|^2 \le \left\|\frac{\partial^7\psi_1(y,z)}{\partial y^4\partial z^3}\right\|^2_{L_2(0<y<q,0<z<r)}, \sum_{m=1}^{+\infty}\frac{1}{m^2} = \frac{\pi^2}{6}.$$

Consequently, the series of the corresponding functions $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ converges absolutely and uniformly. The absolute and uniform convergence of the partial derivative with respect to the variable x up to the third order of the series (21) follows from $\left|\frac{\partial^3 u}{\partial x^3}\right| \le \left|\frac{\partial^2 u}{\partial y^2}\right| + \left|\frac{\partial^2 u}{\partial z^2}\right|$ and what was proved above.

If the domain D^- is considered, that is, the Problem 2.2, then $\lim\limits_{x\to-\infty} X_{n,m}(x) = 0$. Therefore, in (17), there must be $C_{1n,m} = 0$. Then, the function (17) has the form

$$X_{n,m}(x) = e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\frac{\sqrt{3}}{2}k_{n,m}x + C_{3n,m}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right). \tag{24}$$

By virtue of (9), we search for the solution to the Problem 2.2 in the form

$$u(x,y,z)=\sum_{n,m=1}^{+\infty} e^{\frac{1}{2}k_{n,m}x}\left[C_{2n,m}\cos\frac{\sqrt{3}}{2}k_{n,m}x+C_{3n,m}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right]V_{n,m}(y,z). \tag{25}$$

Requiring the series (25) to satisfy the boundary conditions (5), we obtain

$$u(0,y,z)=\psi_2(y,z)=\sum_{m,n=1}^{+\infty} C_{2n,m}V_{n,m}(y,z),$$

$$u_x(0,y,z)=\psi_3(y,z)=\sum_{m,n=1}^{+\infty}\frac{1}{2}k_{n,m}\left[C_{2n,m}+\sqrt{3}C_{3n,m}\right]V_{n,m}(y,z),$$

where

$$\left.\begin{aligned}
&C_{3n,m}=\frac{2}{\sqrt{3}k_{n,m}}\psi_{3n,m}-\frac{1}{\sqrt{3}}\psi_{2n,m},\\
&\psi_{in,m}=\frac{1}{\left\|V_{n,m}\right\|^2}\int_0^q\int_0^r\psi_i(y,z)\,Y_n(y)\sin\frac{m\pi z}{r}dydz, i=2,3,
\end{aligned}\right\} \tag{26}$$

$\psi_{2n,m}$ and $\psi_{3n,m}$ - the Fourier coefficients of the functions $\psi_2(y,z)$ and $\psi_3(y,z)$.

Putting the values $C_{2n,m}$ and $C_{3n,m}$ in the series (25), we get

$$u(x,y,z)=\sum_{n,m=1}^{+\infty} e^{\frac{1}{2}k_{n,m}x}\left[\psi_{1n,m}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x+\frac{\pi}{6}\right)+\psi_{2n,m}\frac{2}{\sqrt{3}k_{n,m}}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right]V_{n,m}(y,z). \tag{27}$$

Now, we prove that the series (27) and its derivatives u_{xxx}, u_{yy} and u_{zz} converge uniformly in the domain $D^-\cup\Gamma_2$, then function $u(x,y,z)$, defined by the series (27), gives a solution to the Problem 2.2.

Let us prove the absolute and uniform convergence of the series (27). From (27) we have

$$\begin{aligned}
\left|u(x,y,z)\right|=\sum_{n,m=1}^{+\infty}\left|e^{\frac{1}{2}k_{n,m}x}\left[\psi_{2n,m}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x+\frac{\pi}{6}\right)+\psi_{3n,m}\frac{2}{\sqrt{3}k_{n,m}}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right]V_{n,m}(y,z)\right|\le\\
\le M\sum_{n,m=1}^{+\infty}\left[\left|\psi_{2n,m}Y_n(y)\right|+\frac{1}{k_{n,m}}\left|\psi_{3n,m}Y_n(y)\right|\right].
\end{aligned} \tag{28}$$

Integrating (26) by parts, taking into account condition (6) and $k_{n,m}=\sqrt[3]{\lambda_{n,m}}=\sqrt[3]{\nu_n+\mu_m}\ge\sqrt[3]{\frac{2\pi^2}{qr}}\sqrt[3]{nm}$, we obtain

$$\left|\psi_{2n,m}Y_n(y)\right|\le M\frac{\left|\psi^{(7)}_{2n,m}\right|}{n^4m^3},\ \left|\psi_{3n,m}Y_n(y)\right|\le M\frac{\left|\psi^{(7)}_{3n,m}\right|}{n^{\frac{13}{3}}m^{\frac{10}{3}}}. \tag{29}$$

Taking these estimates into account, from (28) we have

$$\left|u(x,y,z)\right|=M\left(\sum_{n,m=1}^{+\infty}\frac{\left|\psi^{(7)}_{2n,m}\right|}{n^4m^3}+\sum_{n,m=1}^{+\infty}\frac{\left|\psi^{(7)}_{3n,m}\right|}{n^{\frac{13}{3}}m^{\frac{10}{3}}}\right)<\infty,$$

It follows that the series (27) converges absolutely and uniformly.

Now, we prove that the derivatives of the series (27) included in equation (1) also converge absolutely and uniformly in the domain $D^+ \cup \Gamma_1$. To do this, from (27), we calculate the derivatives by y and z, and we attain

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{n,m=1}^{+\infty} \nu_n e^{\frac{1}{2}k_{n,m}x}\left[\psi_{1n,m}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x+\frac{\pi}{6}\right)+\psi_{2n,m}\frac{2}{\sqrt{3}k_{n,m}}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right]V_{n,m}(y,z),$$

$$\frac{\partial^2 u}{\partial z^2} = -\sum_{n,m=1}^{+\infty} m^2 e^{\frac{1}{2}k_{n,m}x}\left[\psi_{2n,m}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x+\frac{\pi}{6}\right)+\psi_{3n,m}\frac{2}{\sqrt{3}k_{n,m}}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right]V_{n,m}(y,z).$$

Let us estimate the resulting equalities and, taking into account (29), for $x < 0$, we have

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M\left(\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{2n,m}^{(7)}\right|}{n^2 m^3}+\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{3n,m}^{(7)}\right|}{n^{\frac{7}{3}} m^{\frac{10}{3}}}\right) < \infty,$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\left(\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{2n,m}^{(7)}\right|}{n^4 m}+\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{3n,m}^{(7)}\right|}{n^{\frac{13}{3}} m^{\frac{4}{3}}}\right),$$

Using the Cauchy-Bunyakovsky and Bessel inequality, we obtain

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\left(\sqrt{\sum_{m,n=1}^{+\infty}\left|\psi_{2n,m}^{(7)}\right|^2}\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n^4 m}\right)^2}+\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{3n,m}^{(7)}\right|}{n^{\frac{13}{3}} m^{\frac{4}{3}}}\right) \le M\left(\left\|\frac{\partial^7\psi_2(y,z)}{\partial y^4\partial z^3}\right\|_{L_2(0<y<q,0<z<r)}+\sum_{n,m=1}^{+\infty}\frac{\left|\psi_{3n,m}^{(7)}\right|}{n^{\frac{13}{3}} m^{\frac{4}{3}}}\right) < \infty,$$

where

$$\sum_{n,m=1}^{+\infty}\left|\psi_{2n,m}^{(7)}\right|^2 \le \left\|\frac{\partial^7\psi_2(y,z)}{\partial y^4\partial z^3}\right\|^2_{L_2(0<y<q,0<z<r)},\ \sum_{m=1}^{+\infty}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

Consequently, the series of the corresponding functions $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ converges absolutely and uniformly. The absolute and uniform convergence of the partial derivative with respect to the variable x up to the third order of the series (27) follows from $\left|\frac{\partial^3 u}{\partial x^3}\right| \le \left|\frac{\partial^2 u}{\partial y^2}\right| + \left|\frac{\partial^2 u}{\partial z^2}\right|$ and what was proved above.

The proof of Theorem 4.1 is complete. □

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