Filomat 39:9 (2025), 2831–2843
https://doi.org/10.2298/FIL2509831S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the spectral ν-continuity

S. Sánchez-Perales[a], S. Palafox[a], T. Pérez-Becerra[a,*]

[a] *Universidad Tecnológica de la Mixteca, Centro de Modelación Matemática, Vinculación y Consultoría, 69004 Oaxaca, México*

Abstract. The spectrum of elements of Banach algebras can be considered as an application that has as codomain the space of compact subsets of $\mathbb{C}$ with the Hausdorff metric. The essential question to answer, which opens this line of research, is whether this mapping is continuous, on this line, this paper studies whether some subsets of the spectrum are ν-continuous at certain operators. Conceptualizing: the spectrum σ is called ν-continuous if a sequence (T_n) is ν-convergent to T implies $\sigma(T_n) \to \sigma(T)$ in the Hausdorff metric. However, the results of the present paper demonstrate that special care is required about the zero point in the spectrum to guarantee this convergence, which is why additional conditions are requisite on this singular point. For example, it is known that the spectrum is upper semi-ν-continuous, this paper shows that the spectrum is also lower semi-ν-continuous (hence ν-continuous) at a Fredholm operator for which 0 is an accumulation point of the spectrum, and which satisfies a condition on the spectrum similar to one imposed by Conway and Morrey. In addition, in this manuscript, it is established that the approximate point spectrum σ_{ap} is upper semi-ν-continuous except possibly for the zero point in the spectrum and shows the conditions on a Fredholm operator to ensure the approximate point spectrum σ_{ap} is ν-continuous. Finally, it is shown that the lower semi-ν-continuity of the Weyl spectrum can be obtained by restricting to essentially G_1 operators and if a sequence of p-hyponormal operators T_n (which are uniformly bounded below on the complement of the kernel) ν-converges to a Fredholm operator T (for which $0 \in \sigma_{ap}(T)$), then $\sigma(T_n)$ converges to $\sigma(T)$.

1. Introduction

Spectral continuity is a relevant subject in Banach-space theory and operator theory. Conway and Morrey in [7] studied the continuity of the spectrum for bounded operators on Hilbert spaces, in particular, they are looking for the point of continuity and affirm that it would be interesting to extend the fundamental theorem of their investigation to Banach spaces. Several authors have studied this topic in Banach spaces using different types of convergence; in particular, Ahues in [1] proves that the norm convergence implies the upper and lower semi-continuity of the spectrum at each isolated point.

Convergence in norm is not sufficient to approximate an operator, for example, Nistrom in [1, Proposition 4.6] provides an approximation of a Fredholm integral operator on $C^0([a,b])$, which is not norm convergent.

2020 *Mathematics Subject Classification*. Primary 47A10; Secondary 47A53.
Keywords. ν-continuity, Fredholm operator, Spectrum.
Received: 02 April 2024; Accepted: 17 January 2025
Communicated by Dragan S. Djordjević
* Corresponding author: T. Pérez-Becerra
Email addresses: ssanchez@mixteco.utm.mx (S. Sánchez-Perales), palafox@mixteco.utm.mx (S. Palafox), tompb55@hotmail.com (T. Pérez-Becerra)
ORCID iDs: https://orcid.org/0000-0002-0962-2119 (S. Sánchez-Perales), https://orcid.org/0000-0002-4961-4834 (S. Palafox), https://orcid.org/0000-0002-8821-4040 (T. Pérez-Becerra)

The norm convergence condition can be weakened by imposing collectively compact convergence, which is a characteristic that cannot be omitted in this case. Dealing with this problem, Ahues introduces a "new" mode of convergence that does not require compactness: the ν-convergence. In Ahues' paper appears several useful examples of ν-convergence, in particular: (1) the norm convergence, and the collectively compact convergence, imply the ν-convergence; (2) the convergence of a modification of the Nyström approximation is now guaranteed; (3) the operation of addition in the space of bounded operators over a Banach space ($B(X)$) is not compatible with ν-convergence, that is, if $T_n \xrightarrow{\nu} T$ and $U_n \xrightarrow{\nu} U$, it is possible that $T_n + U_n \overset{\nu}{\nrightarrow} T + U$; and (4) the ν-convergence is a pseudo-convergence in the sense that can be possible $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} U$ but $T \neq U$, even so, $\sigma(T) = \sigma(U)$. Time after, Sánchez et al. in [17] establish that problem (4) also occurs for the point spectrum and the approximate point spectrum. Also, Sánchez et al. show that Riesz operators and shift isometries are points of the spectral ν-continuity and investigate the stability of points of spectral continuity for a sequence of compact operators that ν-converges to a Riesz operator.

Ammar in [3] treats the problem (4) and establishes that the Wolf and Weyl essential spectra of T and U are equal, then he inspects the relationship between the Wolf and the Weyl essential spectra of T_n and T for T_n being ν-convergent to T. Moreover, he proves if T is a bounded Fredholm linear operator, then it is ν-upper semi-continuous at T.

In this paper, in Section 2, we study the ν-continuity of the spectral function over arbitrary unital Banach algebras and show that if this type of Banach algebra is abelian, then the spectral function is ν-continuous. We also demonstrate the ν-continuity of this function over a certain set $\mathcal{R}_{\mathcal{A}}$ in the algebra $\mathcal{A}$, see (2). In section 3, the study is made on the ν-continuity of the spectrum and of the approximate point spectrum in the algebra of bounded linear operators defined on a Banach space, which is also exhibited that the compact self-adjoint operators are ν-continuous. An interesting result in this section is that for the particular case of a Hilbert space H, the ν-continuity of the spectrum is equivalent under certain conditions to the usual continuity (in norm) of the spectrum; furthermore, it is proved that the approximate point spectrum is upper semi-ν-continuous in every Fredholm operator, and then sufficient conditions are given to guarantee the ν-continuity, this may be possible by the extension of a result of Conway and Morrel from Hilbert spaces to Banach spaces. On the other hand, due to the work of Luecke in [12], it is known that the Weyl spectrum is continuous if applied to essentially G_1 operators; section 4 is dedicated to establishing that, in this case, also the Weyl spectrum is ν-continuous. In this same context, Djordjević in [9] and Hwang in [11] show the spectrum over the class of p-hyponormal operators is continuous, so this article ends with an adaptation of Djordjevic's idea to prove the ν-continuity of the spectrum over this specific class of operators.

2. Spectral continuity on complex Banach algebras

Let $\mathcal{A}$ be a complex Banach algebra with identity $1_{\mathcal{A}}$. For $x \in \mathcal{A}$ the resolvent of x is defined by $\rho(x) = \{\lambda \in \mathbb{C} : x - \lambda 1_{\mathcal{A}} \text{ is invertible in } \mathcal{A}\}$ and the spectrum of x is given by $\sigma(x) = \mathbb{C} \setminus \rho(x)$. The spectral radius $r(x)$ of x is the number $r(x) = \max\{|\lambda| : \lambda \in \sigma(x)\}$ and it holds that $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_n \|x^n\|^{\frac{1}{n}}$.

A sequence (x_n) in $\mathcal{A}$ is said to be $\nu-$convergent to x, denoted by $x_n \xrightarrow{\nu} x$, if $(\|x_n\|)$ is bounded, $\|(x_n - x)x\| \to 0$ and $\|(x_n - x)x_n\| \to 0$. The ν-convergence is a pseudo-convergence in the sense that it is possible to have $x_n \xrightarrow{\nu} x$ and $x_n \xrightarrow{\nu} y$ but $x \neq y$, see for instance [1, Example 1]. The connection between norm convergence and ν-convergence is as follows: if $x_n \xrightarrow{n} x$ then $x_n \xrightarrow{\nu} x$, also, if $x_n \xrightarrow{\nu} x$ and x is right invertible, then $x_n \xrightarrow{n} x$.

A function τ defined on $\mathcal{A}$ whose values are non-empty compact sets in $\mathbb{C}$ is said to be continuous (ν-continuous, respectively) at x, if $\tau(x_n) \to \tau(x)$ with respect to the Hausdorff metric, for all sequence (x_n) in $\mathcal{A}$ such that $x_n \xrightarrow{n} x$ ($x_n \xrightarrow{\nu} x$, respectively). It is clear that if τ is ν-continuous at x, then τ is continuous at x. The function τ is said to be upper semi-continuous (upper semi-ν-continuous, respectively) at x, if $\limsup \tau(x_n) \subseteq \tau(x)$ for all sequence (x_n) in $\mathcal{A}$ such that $x_n \xrightarrow{n} x$ ($x_n \xrightarrow{\nu} x$, respectively). Also, τ is said to be lower semi-continuous (lower semi-ν-continuous, respectively) at x, if $\tau(x) \subseteq \liminf \tau(x_n)$ for all (x_n) in $\mathcal{A}$ such that $x_n \xrightarrow{n} x$ ($x_n \xrightarrow{\nu} x$, respectively).

Remark 2.1. *Let τ be a function defined on $\mathcal{A}$ whose values are non-empty compact sets in $\mathbb{C}$ such that $\tau(y) \subseteq \sigma(y)$ for all $y \in \mathcal{A}$. Then*

1. *τ is continuous at $x \in \mathcal{A}$ if and only if τ is upper and lower semi-continuous at x.*
2. *τ is ν-continuous at $x \in \mathcal{A}$ if and only if τ is upper and lower semi-ν-continuous at x.*

The following theorem is proved in [1, Corollary 2.7].

Theorem 2.2. *For each $x \in \mathcal{A}$, σ is upper semi-ν-continuous at x.*

In the following proposition, we will use the notation $\sigma_{\mathcal{B}}(x)$ for the spectrum of $x \in \mathcal{B}$ concerning a given subalgebra $\mathcal{B}$ of $\mathcal{A}$. A character on an abelian Banach algebra $\mathcal{B}$ is a non-zero homomorphism $\varphi : \mathcal{B} \to \mathbb{C}$. The set of all characters on $\mathcal{B}$ is denoted by $\mathcal{M}(\mathcal{B})$. In [13, Theorems 4 and 6], it is established that if $\mathcal{B}$ is an unital abelian Banach algebra then

1. $\varphi(x) \in \sigma_{\mathcal{B}}(x)$ for all $x \in \mathcal{B}$ and $\varphi \in \mathcal{M}(\mathcal{B})$;
2. for every $\lambda \in \sigma_{\mathcal{B}}(x)$, there exists $\varphi \in \mathcal{M}(\mathcal{B})$ such that $\varphi(x) = \lambda$;
3. for each $\varphi \in \mathcal{M}(\mathcal{B})$, $\|\varphi\| = 1$.

In [15, Theorem 4], Newburgh proves that commutativity implies the spectral continuity: if the elements of a sequence (x_n) in a Banach algebra $\mathcal{A}$ commute with their limit, i.e. if $x_n \xrightarrow{n} x$ and $x_n x = x x_n$, then $\sigma(x_n) \to \sigma(x)$. The following proposition gives an essential condition to extend that result for the ν-convergence: 0 must be an accumulation point in the spectrum of a.

Proposition 2.3. *Let $a \in \mathcal{A}$ and (a_n) be a sequence in $\mathcal{A}$ such that $a_n a = a a_n$ and $a_n a_m = a_m a_n$ for all $n, m \in \mathbb{N}$. If $a_n \xrightarrow{\nu} a$ and $0 \in \operatorname{acc} \sigma(a)$, then $\sigma(a_n) \to \sigma(a)$.*

Proof. By Theorem 2.2, σ is upper semi-ν-continuous at a, thus $\limsup \sigma_{\mathcal{A}}(a_n) \subseteq \sigma_{\mathcal{A}}(a)$. Hence, we only need to prove that $\sigma_{\mathcal{A}}(a) \subseteq \liminf \sigma_{\mathcal{A}}(a_n)$.

Consider $\mathcal{B}_0$ the subalgebra of $\mathcal{A}$ which consists of all linear combinations of finite products of elements in $\{a_n : n \in \mathbb{N}\} \cup \{a, 1_{\mathcal{A}}\}$. From hypothesis, $\mathcal{B}_0$ is commutative. Thus by Zorn's lemma, there exists $\mathcal{B}$ the maximal abelian subalgebra of $\mathcal{A}$ such that $\mathcal{B}_0 \subseteq \mathcal{B}$. Therefore by [14, Exercise 8, p.8], $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ and $\sigma_{\mathcal{A}}(a_n) = \sigma_{\mathcal{B}}(a_n)$ for all $n \in \mathbb{N}$. Let $\lambda \in \sigma_{\mathcal{A}}(a)$ with $\lambda \neq 0$. Then there exists $\varphi \in \mathcal{M}(\mathcal{B})$ such that $\varphi(a) = \lambda$. Observe that

$$\|(\varphi(a_n) - \varphi(a))\varphi(a)\| = \|\varphi((a_n - a)a)\| \leq \|\varphi\| \|(a_n - a)a\| \to 0.$$

Thus $\|(\varphi(a_n) - \varphi(a))\lambda\| \to 0$, which implies that $\varphi(a_n) \to \lambda$. Now, since $\varphi(a_n) \in \sigma_{\mathcal{B}}(a_n)$ ($= \sigma_{\mathcal{A}}(a_n)$) for all $n \in \mathbb{N}$, it follows that $\lambda \in \liminf \sigma_{\mathcal{A}}(a_n)$. Consequently, since $0 \in \operatorname{acc} \sigma_{\mathcal{A}}(a)$,

$$\sigma_{\mathcal{A}}(a) = \overline{\sigma_{\mathcal{A}}(a) \setminus \{0\}} \subseteq \liminf \sigma_{\mathcal{A}}(a_n).$$

□

Remember, an elementary Cauchy domain is an open bounded connected subset of $\mathbb{C}$ whose boundary is the union of a finite number of nonintersecting Jordan curves. A finite union of elementary Cauchy domains having disjoint closures is called a Cauchy domain. Let D be a Cauchy domain, if each curve involved in the boundary of D is oriented in such a way that points in D lie to the left as the curve is traced out, then the oriented boundary C of D is called a Cauchy contour. The interior of the Cauchy contour C is defined as $\operatorname{int}(C) = D$ and the exterior of C is defined by $\operatorname{ext}(C) = \mathbb{C} \setminus (D \cup C)$.

A set $\Lambda \subseteq \sigma(a)$ is a spectral set for a if Λ is closed and open in $\sigma(a)$. We set $\mathcal{C}(a, \Lambda)$ the set of all Cauchy contours C separating Λ from $\sigma(a) \setminus \Lambda$, i.e. $\Lambda \subseteq \operatorname{int}(C)$ and $\sigma(a) \setminus \Lambda \subseteq \operatorname{ext}(C)$. It is clear that if $C \in \mathcal{C}(a, \Lambda)$, then $C \subseteq \rho(a)$. For any $a \in \mathcal{A}$, Λ a spectral set for a and $C \in \mathcal{C}(a, \Lambda)$, define

$$p(a, \Lambda) = -\frac{1}{2\pi i} \int_C (a - z)^{-1} dz.$$

The element $p(a, \Lambda)$ does not depend on the choice of $C \in \mathcal{C}(a, \Lambda)$.

Remark 2.4. *Let $a \in \mathcal{A}$, Λ a spectral set for a and $C \in \mathcal{C}(a,\Lambda)$. If $p = p(a,\Lambda)$, then*

1. $p^2 = p$ *and* $pa = ap$;
2. $\Lambda = \emptyset$ *if and only if* $p(a,\Lambda) = 0$.

Proposition 2.5. *Let $p, q \in \mathcal{A}$ be such that $p^2 = p$. If $p \neq 0$ and $r(p-q) < 1$ then $q \neq 0$.*

Proof. Since $r(p-q) < 1$ it follows that $(p-q) - 1_{\mathcal{A}}$ is invertible. Suppose that $q = 0$, then $p - 1_{\mathcal{A}}$ is invertible. Thus there exists $z \in \mathcal{A}$ such that $(p - 1_{\mathcal{A}})z = 1_{\mathcal{A}}$. This implies that $p(p-1_{\mathcal{A}})z = p1_{\mathcal{A}}$ and so $(p^2 - p)z = p$, i.e. $0 = (p-p)z = p$, which is a contradiction. □

Proposition 2.6. *If $a, b \in \mathcal{A}$ and $\lambda \in \rho(b)$ with $\lambda \neq 0$, then*

$$[(a-b)(b-\lambda 1_{\mathcal{A}})^{-1}]^2 = \frac{1}{\lambda}\Big[(a-b)b(b-\lambda 1_{\mathcal{A}})^{-1}(a-b) - (a-b)a + (a-b)b\Big](b-\lambda 1_{\mathcal{A}})^{-1}.$$

Proof. Since $(b-\lambda 1_{\mathcal{A}})(b-\lambda 1_{\mathcal{A}})^{-1} = 1$ it follows that $b(b-\lambda 1_{\mathcal{A}})^{-1} - \lambda(b-\lambda 1_{\mathcal{A}})^{-1} = 1_{\mathcal{A}}$. Therefore $(b-\lambda 1_{\mathcal{A}})^{-1} = \frac{1}{\lambda}[b(b-\lambda 1_{\mathcal{A}})^{-1} - 1_{\mathcal{A}}]$. Thus,

$$\begin{aligned}
& [(a-b)(b-\lambda 1_{\mathcal{A}})^{-1}]^2 \\
= \; & (a-b)(b-\lambda 1_{\mathcal{A}})^{-1}(a-b)(b-\lambda 1_{\mathcal{A}})^{-1} \\
= \; & (a-b)\frac{1}{\lambda}\Big[b(b-\lambda 1_{\mathcal{A}})^{-1} - 1_{\mathcal{A}}\Big](a-b)(b-\lambda 1_{\mathcal{A}})^{-1} \\
= \; & \frac{1}{\lambda}\Big[(a-b)b(b-\lambda 1_{\mathcal{A}})^{-1}(a-b) - (a-b)a + (a-b)b\Big](b-\lambda 1_{\mathcal{A}})^{-1}.
\end{aligned}$$

□

Proposition 2.7. *Let $a, b \in \mathcal{A}$. If $\lambda \in \rho(b)$ and $\|[(a-b)(b-\lambda 1_{\mathcal{A}})^{-1}]^2\| < 1$, then $\lambda \in \rho(a)$.*

Proof. By the spectral radius theorem, $r\Big((a-b)(b-\lambda 1_{\mathcal{A}})^{-1}\Big) \leq \|[(a-b)(b-\lambda 1_{\mathcal{A}})^{-1}]^2\|^{\frac{1}{2}} < 1$ and so $-1 \in \rho\Big((a-b)(b-\lambda 1_{\mathcal{A}})^{-1}\Big)$. Then $(a-b)(b-\lambda 1_{\mathcal{A}})^{-1} + 1_{\mathcal{A}}$ is invertible. Therefore

$$a - \lambda 1_{\mathcal{A}} = a - b + b - \lambda 1_{\mathcal{A}} = \Big[(a-b)(b-\lambda 1_{\mathcal{A}})^{-1} + 1_{\mathcal{A}}\Big](b-\lambda 1_{\mathcal{A}})$$

is invertible. □

Theorem 2.8. *[1, Proposition 2.9] Let $a \in \mathcal{A}$, Λ be a spectral set for a, $C \in \mathcal{C}(a,\Lambda)$ and (a_n) be a sequence in $\mathcal{A}$ such that $a_n \xrightarrow{\nu} a$. Then*

1. *There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, C lies in $\rho(a_n)$.*
2. *If $\Lambda_n := \sigma(a_n) \cap \mathrm{int}(C)$ for all $n \geq n_0$, then Λ_n is a spectral set for a_n and $C \in \mathcal{C}(a_n,\Lambda_n)$. Further, if $0 \in \mathrm{ext}(C)$ then $p(a_n,\Lambda_n) \xrightarrow{\nu} p(a,\Lambda)$.*

Conway and Morrey prove the following lemma, widely used in spectral approximation under norm convergence in Hilbert spaces ([7, Lemma 1.5]). They state that the proof can be found in the literature, but it is unknown for Banach spaces. However, this has been achieved for unital Banach algebras now, utilizing ν−convergence, although the point 0 has been omitted.

Lemma 2.9. *Let $a \in \mathcal{A}$ and (a_n) be a sequence in $\mathcal{A}$ such that $a_n \xrightarrow{\nu} a$. If U is an open set for which $0 \notin U$ and U contains a component of $\sigma(a)$, then there exits $n_0 \in \mathbb{N}$ such that U contains a component of $\sigma(a_n)$ for all $n \geq n_0$.*

Proof. Let Ω be a component of $\sigma(a)$ and U be an open set of $\mathbb{C}$ such that $0 \notin U$ and $\Omega \subseteq U$. Since $\Omega \cap [\sigma(a) \setminus U] = \emptyset$, $\sigma(a) \setminus U$ is closed and $\sigma(a)$ is compact, it follows that there exists $\Lambda \neq \emptyset$ open and closed set in $\sigma(a)$ such that $\Omega \subseteq \Lambda$ and $\Lambda \cap [\sigma(a) \setminus U] = \emptyset$. This implies that $\Lambda \subseteq U$. From [1, Theorem 1.21], there exists a Cauchy domain D such that

$$\Lambda \subset D \quad \text{and} \quad \overline{D} \subset U \cap [\mathbb{C} \setminus (\sigma(a) \setminus \Lambda)]. \tag{1}$$

Let C be the boundary of D oriented in a way that C is a Cauchy contour. It is clear by (1) that $C \in C(a, \Lambda)$. Then from Theorem 2.8, there exits $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, C lies in $\rho(a_n)$. Further, if $\Lambda_n := \sigma(a_n) \cap D$ for all $n \geq n_1$ then since $0 \notin U$ we have that

$$[r(p - p_n)]^2 \leq \|(p - p_n)^2\| \leq \|(p_n - p)p_n\| + \|(p_n - p)p\| \to 0,$$

where $p_n = p(a_n, \Lambda_n)$ and $p = p(a, \Lambda)$. Thus there exists $n_0 \in \mathbb{N}$ with $n_0 \geq n_1$ such that $r(p - p_n) < 1$ for all $n \geq n_0$. Since $\Lambda \neq \emptyset$ we have from Remark 2.4 that $p \neq 0$. Therefore, by Proposition 2.5, $p_n \neq 0$ for all $n \geq n_0$. Thus by Remark 2.4, $\Lambda_n \neq \emptyset$ for all $n \geq n_0$. This implies, since Λ_n is both open and closed in $\sigma(a_n)$, that there exists Ω_n a component of $\sigma(a_n)$ such that $\Omega_n \subseteq \Lambda_n$. Observe that $\Lambda_n \subseteq D \subseteq U$. Thus $\Omega_n \subseteq U$. Therefore U contains a component of $\sigma(a_n)$ for all $n \geq n_0$. □

Remark 2.10. *From Lemma 2.9 we have that if $\lambda \in \operatorname{iso} \sigma(a)$ with $\lambda \neq 0$, then $\lambda \in \liminf \sigma(a_n)$ for all sequence (a_n) in $\mathcal{A}$ such that $a_n \xrightarrow{\nu} a$.*

We define

$$\mathcal{R}_{\mathcal{A}} = \left\{a \in \mathcal{A} : \ \|(a - \lambda 1_{\mathcal{A}})^{-1}\| = r((a - \lambda 1_{\mathcal{A}})^{-1}) \text{ for all } \lambda \in \rho(a)\right\}. \tag{2}$$

If H is a Hilbert space and $\mathcal{A} = B(H)$, then normal, subnormal, and hyponormal operators are elements of $\mathcal{R}_{\mathcal{A}}$.

Theorem 2.11. *Let $a \in \mathcal{A}$ and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}_{\mathcal{A}}$ such that $a_n \xrightarrow{\nu} a$. If 0 is an accumulation point of $\sigma(a)$, then $\sigma(a_n) \to \sigma(a)$.*

Proof. Let $\lambda \in \sigma(a) \setminus \{0\}$. Suppose that $\lambda \notin \liminf \sigma(a_n)$. Then there exist $\epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(a_n) = \emptyset$ for infinite number of n's. Without loss of generality assume that this holds for all n. This implies that $\epsilon < \mathrm{d}(\lambda, \sigma(a_n)) = \mathrm{d}(0, \sigma(a_n - \lambda 1_{\mathcal{A}}))$ and so $0 \notin \sigma(a_n - \lambda 1_{\mathcal{A}})$. By the spectral mapping theorem,

$$\sigma\left((a_n - \lambda 1_{\mathcal{A}})^{-1}\right) = \{\mu^{-1} \ : \ \mu \in \sigma(a_n - \lambda 1_{\mathcal{A}})\}.$$

Therefore,

$$\begin{aligned}\|(a_n - \lambda 1_{\mathcal{A}})^{-1}\| &= r((a_n - \lambda 1_{\mathcal{A}})^{-1}) = \max\{|\mu|^{-1} \ : \ \mu \in \sigma(a_n - \lambda 1_{\mathcal{A}})\} \\ &= \frac{1}{\min\{|\mu| \ : \ \mu \in \sigma(a_n - \lambda 1_{\mathcal{A}})\}} = \frac{1}{\mathrm{d}(0, \sigma(a_n - \lambda 1_{\mathcal{A}}))} < \frac{1}{\epsilon}.\end{aligned}$$

Thus by Proposition 2.6, $\|[(a-a_n)(a_n-\lambda 1_{\mathcal{A}})^{-1}]^2\| \leq \frac{1}{|\lambda|}\Big[\|(a-a_n)a_n\|\frac{1}{\epsilon}(\|a\|+\|a_n\|)+\|(a-a_n)a\|+\|(a-a_n)a_n\|\Big]\frac{1}{\epsilon}$. Now, since $a_n \xrightarrow{\nu} a$, it follows that $\|(a - a_n)a\| \to 0$, $\|(a - a_n)a_n\| \to 0$ and $\{\|a_n\|\}_{n\in\mathbb{N}}$ is bounded. Therefore, $\|[(a - a_n)(a_n - \lambda 1_{\mathcal{A}})^{-1}]^2\| \to 0$. Then there exists $n^* \in \mathbb{N}$ such that $\|[(a - a_{n^*})(a_{n^*} - \lambda 1_{\mathcal{A}})^{-1}]^2\| < 1$.

Consequently, by Propposition 2.7, $\lambda \in \rho(a)$, which is a contradiction. Therefore $\lambda \in \liminf \sigma(a_n)$.

Now, since 0 is an accumulation point of $\sigma(a)$, it follows that $\sigma(a) = \overline{\sigma(a) \setminus \{0\}} \subseteq \overline{\liminf \sigma(a_n)} \subseteq \liminf \sigma(a_n)$. □

3. Spectral continuity in the algebra $B(X)$

Let X be a Banach space and $B(X)$ be the algebra of all bounded linear operators defined on X. For $T \in B(X)$, let $N(T)$ and $R(T)$ denote the null space and the range of the mapping T. Let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$, if these spaces are finite dimensional, otherwise let $\alpha(T) = \infty$ and $\beta(T) = \infty$. If the range $R(T)$ of $T \in B(X)$ is closed and $\alpha(T) < \infty$, then T is said to be an *upper semi-Fredholm* operator ($T \in \Phi_+(X)$). Similarly, if $\beta(T) < \infty$, then T is said to be a *lower semi-Fredholm* operator ($T \in \Phi_-(X)$). If $T \in \Phi_-(X) \cup \Phi_+(X)$ then T is called a *semi-Fredholm* operator ($T \in \Phi_\pm(X)$) and for $T \in \Phi_-(X) \cap \Phi_+(X)$ we say that T is a *Fredholm* operator ($T \in \Phi(X)$). For $T \in \Phi_\pm(X)$, the *index* of T is defined by $i(T) = \alpha(T) - \beta(T)$. It is well known that the index is a continuous function on the set of semi-Fredholm operators. This property also holds for the ν-convergence. See [17, Theorem 3.4].

For $T \in B(X)$ let $\rho_{sf}(T)$ be denote the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \in \Phi_\pm(X)$, and for $k \in \mathbb{Z} \cup \{-\infty, \infty\}$, let $\rho_{sf}^k(T)$ be the set of $\lambda \in \rho_{sf}(T)$ such that $i(T - \lambda) = k$. Put

$$\rho_{sf}^-(T) = \bigcup_{-\infty \le k \le -1} \rho_{sf}^k(T), \quad \rho_{sf}^+(T) = \bigcup_{1 \le k \le \infty} \rho_{sf}^k(T), \quad \rho_{sf}^\pm(T) = \rho_{sf}^-(T) \cup \rho_{sf}^+(T).$$

The semi-Fredholm spectrum, the approximate point spectrum, the surjective spectrum, the point spectrum, the Weyl spectrum, and the set Riesz points of $T \in B(X)$ are defined by $\sigma_{sf}(T) = \mathbb{C} \setminus \rho_{sf}(T)$, $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$, $\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}$, $\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}$, $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator of index zero}\}$ and $\pi_0(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an isolated eigenvalue of } T \text{ of finite algebraic multiplicity}\}$, respectively.

Let $K(X)$ denote the set of all compact linear operators in $B(X)$. If $\pi : B(X) \to B(X)/K(X)$ is the canonical homomorphism, then the essential spectrum of an operator $T \in B(X)$, $\sigma_e(T)$, is the spectrum of $\pi(T)$ in the Calkin algebra $B(X)/K(X)$.

An operator $T \in B(X)$ is said that satisfies Browder's theorem if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T).$$

Remark 3.1. *Let $T \in B(X)$ and (T_n) be a sequence in $B(X)$ such that $T_n \xrightarrow{\nu} T$. The following inclusions are holding:*

1. $\pi_0(T) \subseteq \liminf \pi_0(T_n)$. *See, [1, Corollary 2.13].*
2. $[\operatorname{iso} \sigma(T)] \setminus \{0\} \subseteq \liminf \sigma(T_n)$. *See Remark 2.10.*

Moreover, if $T \in \Phi(X)$ then

3. $\overline{\rho_{sf}^+(T)} \subseteq \liminf \sigma_{ap}(T_n)$. *See, [17, Theorem 3.6].*
4. $\overline{\rho_{sf}^-(T)} \subseteq \liminf \sigma_s(T_n)$.
5. $\overline{\rho_{sf}^\pm(T)} \subseteq \liminf \sigma_w(T_n)$.

Proposition 3.2. *Let $T \in \Phi(X)$ and (T_n) be a sequence in $B(X)$ such that $T_n \xrightarrow{\nu} T$. If $\Gamma = \Big\{\lambda \in \sigma(T) :$ for every $\epsilon > 0$, there exist points $\mu_1, \mu_2 \in B(\lambda, \epsilon)$ such that $T - \mu_1, T - \mu_2 \in \Phi_\pm(X)$ and $i(T - \mu_1) \neq i(T - \mu_2)\Big\}$, then $\Gamma \subseteq \liminf \sigma_{sf}(T_n)$*

Proof. Let $\lambda \in \Gamma$ and $\epsilon > 0$. Then there exist $\mu_1, \mu_2 \in B(\lambda, \epsilon)$ such that $T - \mu_1, T - \mu_2 \in \Phi_\pm(X)$ and $i(T - \mu_1) \neq i(T - \mu_2)$. Since $T_n \xrightarrow{\nu} T$, it follows by [17, Theorem 3.4], that there exists $N \in \mathbb{N}$ such that $i(T_n - \mu_i) = i(T - \mu_i)$ for $i = 1, 2$ and for all $n \ge N$. Let $\Lambda = \{\mu_1 + t(\mu_2 - \mu_1) : t \in [0, 1]\}$. Take $n \ge N$ and suppose that for every $\gamma \in \Lambda$, $T_n - \gamma \in \Phi_\pm(X)$. By the continuity of the index, it follows that for every $\gamma \in \Lambda$, there exists $r_\gamma > 0$ such that $i(T_n - z) = i(T_n - \gamma)$ for all $z \in B(\gamma, r)$. The compactness of Λ implies the existence of $\gamma_1, \cdots, \gamma_m$ such that $\Lambda \subseteq \bigcup_{k=1}^{m} B(\gamma_k, r_k)$. Then $i(T - \mu_1) = i(T - \mu_2)$, which is a contradiction. Therefore there exists $\gamma^* \in \Lambda$ such that $T_n - \gamma^* \notin \Phi_\pm(X)$. Thus $B(\lambda, \epsilon) \cap \sigma_{sf}(T_n) \neq \emptyset$. Consequently, $\lambda \in \liminf \sigma_{sf}(T_n)$. □

A classical characterization of the continuity of the spectrum in Hilbert spaces given by [7, Theorem 3.1] is the following: σ is continuous at T if and only if for each $\lambda \in \sigma(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T) \cup \pi_0(T)$. The condition on the right along with two additional ones implies the ν-continuity of the spectrum:

Theorem 3.3. *Let $T \in B(X)$ be such that $0 \in \operatorname{acc}\sigma(T)$ and for each $\lambda \in \sigma(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T) \cup \pi_0(T)$. If one of the following conditions holds*

1. $\rho_{sf}^{\pm}(T) = \emptyset$;
2. $T \in \Phi(X)$;

then σ is ν-continuous at T.

Proof. First observe that $\sigma(T) \setminus \sigma_w(T) = \pi_0(T) \cup \operatorname{int}[\sigma(T) \setminus \sigma_w(T)]$. If we suppose that $\operatorname{int}[\sigma(T) \setminus \sigma_w(T)] \neq \emptyset$, then there exist $\lambda \in \mathbb{C}$ and $r > 0$ such that $B(\lambda, r) \subseteq \operatorname{int}[\sigma(T) \setminus \sigma_w(T)]$. This implies that $\lambda \notin \overline{\rho_{sf}^{\pm}(T)}$ thus from hypothesis $B(\lambda, r)$ contains a component C of $\sigma_{sf}(T) \cup \pi_0(T)$. Consequently,

$$\emptyset \neq C \subseteq \operatorname{int}[\sigma(T) \setminus \sigma_w(T)] \cap [\sigma_{sf}(T) \cup \pi_0(T)],$$

which is a contradiction. Therefore $\operatorname{int}[\sigma(T) \setminus \sigma_w(T)] = \emptyset$ and so T satisfies Browder's theorem.

Let (T_n) be a sequence in $B(X)$ such that $T_n \xrightarrow{\nu} T$. By Theorem 2.2, σ is upper semi-ν-continuous at T. Let $\lambda \in \sigma(T) \setminus \{0\}$. If $\lambda \notin \overline{\rho_{sf}^{\pm}(T)}$ then there exists $\delta > 0$ such that $B(\lambda, \delta) \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$ and $0 \notin B(\lambda, \delta)$. Let $\epsilon > 0$ with $\epsilon < \delta$, from hypothesis, the ball $B(\lambda, \epsilon)$ contains a component Ω of $\sigma_{sf}(T) \cup \pi_0(T)$. By [16, Lemma 3.6], Ω is a component of $\sigma(T)$. Therefore by Lemma 2.9, there exists $n_0 \in \mathbb{N}$ such that $B(\lambda, \epsilon)$ contains a component Ω_n of $\sigma(T_n)$ for all $n \geq n_0$. Thus $B(\lambda, \epsilon) \cap \sigma(T_n) \neq \emptyset$ for all $n \geq n_0$. Consequently, $\lambda \in \liminf \sigma(T_n)$.

Now, if $\lambda \in \overline{\rho_{sf}^{\pm}(T)}$ then assuming hypothesis (2) we obtain by Remark 3.1 (5) that $\lambda \in \overline{\rho_{sf}^{\pm}(T)} \subseteq \liminf \sigma(T_n)$. Consequently, $\sigma(T) = \overline{\sigma(T) \setminus \{0\}} \subseteq \liminf \sigma(T_n)$. Thus σ is lower semi-ν-continuous at T. □

The self-adjoint operators are not points of ν-continuity of the spectrum, (see [16, Example 2.9]), however by Theorem 3.3, the compact self-adjoint operators are.

Example 3.4. *Let $\alpha_{nk} = (1 - \frac{1}{n})\exp(2\pi i \frac{k}{n})$ for all $n \in \mathbb{N}$ and $1 \leq k \leq n$, and consider $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ the diagonal operator defined by*

$$A = \begin{pmatrix} \alpha_{11} & & & \\ & \alpha_{21} & & \\ & & \alpha_{22} & \\ & & & \ddots \end{pmatrix}.$$

Then $\pi_0(A) = \{\alpha_{nk} \mid n \in \mathbb{N},\ 1 \leq k \leq n\}$, $\sigma(A) = \{\alpha_{nk} \mid n \in \mathbb{N},\ 1 \leq k \leq n\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_{sf}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\rho_{sf}^{\pm}(A) = \emptyset$. Therefore by Theorem 3.3, $\sigma(A) \setminus \{0\} \subseteq \liminf \sigma(A_n) \subseteq \limsup \sigma(A_n) \subseteq \sigma(A)$ for all $A_n \xrightarrow{\nu} A$.

Corollary 3.5. *Let H be a Hilbert space and $T \in B(H)$ be such that $0 \in \operatorname{acc}\sigma(T)$. If one of the following conditions holds*

1. $\rho_{sf}^{\pm}(T) = \emptyset$;
2. $T \in \Phi(H)$;

then, σ is continuous at T if and only if σ is ν-continuous at T.

Proof. It is clear that the ν-continuity of σ at T implies the continuity of σ at T. Now, if σ is continuous at T, then by [7, Theorem 3.1], for each $\lambda \in \sigma(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T) \cup \pi_0(T)$. Therefore, by Theorem 3.3, σ is ν-continuous at T. □

Theorem 3.6. *Let $T \in B(X)$ be such that $T \in \Phi(X)$ or $\rho^-_{sf}(T) = \emptyset$. If (T_n) is a sequence in $B(X)$ such that $T_n \xrightarrow{\nu} T$ then*

$$[\limsup \sigma_{ap}(T_n)] \setminus \{0\} \subseteq \sigma_{ap}(T).$$

Proof. Consider (T_n) a sequence in $B(X)$ which is $\nu-$convergent to T. First we suppose that $\rho^-_{sf}(T) = \emptyset$. Take $\lambda \in [\limsup \sigma_{ap}(T_n)] \setminus \{0\}$, then $\lambda \in \limsup \sigma(T_n)$. From Theorem 2.2, $\lambda \in \sigma(T)$. If $\lambda \notin \sigma_{ap}(T)$ then $R(T-\lambda)$ is closed, $\alpha(T-\lambda) = 0$ and $\beta(T-\lambda) > 0$. Consequently, $i(T-\lambda) < 0$ i.e. $\lambda \in \rho^-_{sf}(T)$ which is a contradiction.

Now, we suppose that $T \in \Phi(X)$. Let D, E be closed subspaces of X with $\dim E < \infty$ such that

$$X = N(T) \oplus D \quad \text{and} \quad X = R(T) \oplus E. \tag{3}$$

Let $\lambda \in \limsup \sigma_{ap}(T_n)$ with $\lambda \neq 0$. There exist an increasing sequence of natural numbers (n_k) and points $\lambda_{n_k} \in \sigma_{ap}(T_{n_k})$ such that $\lambda_{n_k} \to \lambda$. Suppose that $\lambda \notin \sigma_{ap}(T)$. Then $T - \lambda \in \Phi_+(X)$ and $\alpha(T-\lambda) = 0$. By (3), $R(T|_D) = R(T)$ and so $T|_D$ is bounded below, therefore by [18, Theorem 5.26], $(T-\lambda)T|_D \in \Phi_+(D, X)$ and $\alpha((T-\lambda)T|_D) = 0$.

On the other hand, observe that

$$(T_{n_k} - \lambda_{n_k})T|_D = (T-\lambda)T|_D + (T_{n_k} - T)T|_D + (\lambda - \lambda_{n_k})T|_D.$$

From $\|(T_n - T)T\| \to 0$ we have that $(T_{n_k} - \lambda_{n_k})T|_D$ converges in norm to $(T-\lambda)T|_D$. Consequently by [18, Theorem 5.23], there exists $k_0 \in \mathbb{N}$ such that every $k \geq k_0$,

$$(T_{n_k} - \lambda_{n_k})T|_D \in \Phi_+(D, X) \ \text{and} \ \alpha((T_{n_k} - \lambda_{n_k})T|_D) = 0. \tag{4}$$

Suppose that for each $k \geq k_0$,

$$N(T_{n_k} - \lambda_{n_k}) \cap E \neq \{0\}.$$

Take $v_k \in N(T_{n_k} - \lambda_{n_k}) \cap E$ with $\|v_k\| = 1$ for all $k \geq k_0$. Since $\dim E < \infty$ it follows that $F := \{e \in E : \|e\| = 1\}$ is a compact set. Therefore we may assume without loss of generality that there exists $v \in F$ such that $v_k \to v$. Observe that

$$\begin{aligned} \|(T_{n_k} - T)T_{n_k}\| &\geq \|(T_{n_k} - T)T_{n_k} v_k\| \\ &= \|(T_{n_k} - T)\lambda_{n_k} v_k\| \\ &= |\lambda_{n_k}|\|\lambda_{n_k} v_k - T v_k\|, \end{aligned}$$

for all $k \geq k_0$, and $|\lambda_{n_k}|\|\lambda_{n_k} v_k - T v_k\| \to |\lambda|\|\lambda v - T v\|$. This implies that

$$|\lambda|\|\lambda v - Tv\| = \lim |\lambda_{n_k}|\|\lambda_{n_k} v_k - T v_k\| \leq \lim \|(T_{n_k} - T)T_{n_k}\| = 0,$$

and so $\|\lambda v - Tv\| = 0$, i.e. $Tv = \lambda v$. Consequently, $\lambda \in \sigma_p(T) (\subseteq \sigma_{ap}(T))$, which is a contradiction. Therefore there exists $k' \geq k_0$ such that $N(T_{n_{k'}} - \lambda_{n_{k'}}) \cap E = \{0\}$. Now, let $y \in R(T) \cap N(T_{n_{k'}} - \lambda_{n_{k'}})$. There exists $x \in X$ such that $y = Tx$ and by (3) there are $n_x \in N(T)$ and $d_x \in D$ such that $x = n_x + d_x$. Thus $(T_{n_{k'}} - \lambda_{n_{k'}})Td_x = (T_{n_{k'}} - \lambda_{n_{k'}})Tx = (T_{n_{k'}} - \lambda_{n_{k'}})y = 0$. Therefore, by (4), $d_x = 0$ and so $y = 0$. Thus, $R(T) \cap N(T_{n_{k'}} - \lambda_{n_{k'}}) = \{0\}$. Consequently,

$$X = R(T) \oplus N(T_{n_{k'}} - \lambda_{n_{k'}}) \oplus E. \tag{5}$$

Then by (3) and (5),

$$\begin{aligned} \dim E = \dim X/R(T) &= \dim[N(T_{n_{k'}} - \lambda_{n_{k'}}) \oplus E] \\ &= \dim N(T_{n_{k'}} - \lambda_{n_{k'}}) + \dim E. \end{aligned}$$

Hence $\dim N(T_{n_{k'}} - \lambda_{n_{k'}}) = 0$ and so $N(T_{n_{k'}} - \lambda_{n_{k'}}) = \{0\}$. From (5),

$$R(T_{n_{k'}} - \lambda_{n_{k'}}) = (T_{n_{k'}} - \lambda_{n_{k'}})T(D) + (T_{n_{k'}} - \lambda_{n_{k'}})(E),$$

which implies that $R(T_{n_{k'}} - \lambda_{n_{k'}})$ is closed. Therefore $\lambda_{n_{k'}} \notin \sigma_{ap}(T_{n_{k'}})$, a contradiction. Consequently, $\lambda \in \sigma_{ap}(T)$. $\square$

The following proposition is a generalization of [7, Proposition 1.3] to the case of Banach spaces, the proof is similar to the one given by Conway, however, for the sake of completeness, we will give its proof.

Proposition 3.7. *Let $T \in B(X)$. If C is a component of $\sigma_{sf}(T)$ such that $C \cap \overline{\rho^{\pm}_{sf}(T)} = \emptyset$, then C is a component of $\sigma_e(T)$.*

Proof. Let C be a component of $\sigma_{sf}(T)$ such that $C \cap \overline{\rho^{\pm}_{sf}(T)} = \emptyset$. Then there exists $\epsilon_1 > 0$ such that $(C)_{\epsilon_1} \cap \overline{\rho^{\pm}_{sf}(T)} = \emptyset$. Observe that

$$\sigma_e(T) = \sigma_{sf}(T) \cup \overline{\rho^{\pm\infty}_{sf}(T)}. \tag{6}$$

Let D be a component of $\sigma_e(T)$ such that $C \subseteq D$. Then by (6),

$$D = (\sigma_{sf}(T) \cap D) \cup (\overline{\rho^{\pm\infty}_{sf}(T)} \cap D).$$

We set $K = \sigma_{sf}(T) \cap D$ and show that for every $\epsilon > 0$, $K \subseteq (C)_\epsilon$. Suppose the opposite, that there exists $\epsilon_2 > 0$ such that $K \nsubseteq (C)_{\epsilon_2}$. Let $r = \min\{\epsilon_1, \epsilon_2\}$, then

$$(C)_r \cap \overline{\rho^{\pm}_{sf}(T)} = \emptyset \quad \text{and} \quad K \setminus (C)_r \neq \emptyset. \tag{7}$$

Define $\mathcal{U} = \{K \setminus A : A \text{ is open and closed set in } K \text{ with } C \subseteq A\}$. Since quasi-components and components coincide for compact Hausdorff spaces, it follows that $\mathcal{U}$ is an open cover of $K \setminus (C)_r$. By the compactness of $K \setminus (C)_r$, there exist $A_1, \cdots, A_n$ open and closed sets in K with $C \subseteq A_i$, $i = 1 \cdots, n$, such that

$$K \setminus (C)_r \subseteq \bigcup_{i=1}^{n} (K \setminus A_i) = K \setminus \bigcap_{i=1}^{n} A_i. \tag{8}$$

We set $A_0 = \bigcap_{i=1}^{n} A_i$ and $B_0 = K \setminus A_0$. Then A_0, B_0 are compact sets such that $K = A_0 \cup B_0$, $A_0 \cap B_0 = \emptyset$, $C \subseteq A_0 \subseteq (C)_r$, $A_0 \neq \emptyset$, and by (7) and (8), $B_0 \neq \emptyset$. Therefore

$$D = A_0 \cup [B_0 \cup (\overline{\rho^{\pm\infty}_{sf}(T)} \cap D)] \quad \text{and} \quad A_0 \cap [B_0 \cup (\overline{\rho^{\pm\infty}_{sf}(T)} \cap D)] = \emptyset.$$

This implies that D is disconnected, which is a contradiction. Thus for every $\epsilon > 0$, $C \subseteq K \subseteq (C)_\epsilon$. Then K is connected, and so $K = C$. Consequently, $D = C \cup (\overline{\rho^{\pm\infty}_{sf}(T)} \cap D)$. Since D is connected, $C \neq \emptyset$ and $C \cap (\overline{\rho^{\pm\infty}_{sf}(T)} \cap D) = \emptyset$, it follows that $\overline{\rho^{\pm\infty}_{sf}(T)} \cap D = \emptyset$. Therefore $D = C$. □

Conway and Morrel in [8, Theorem 5.1] characterized the continuity of the approximate point spectrum in Hilbert spaces through the following conditions:

1. For each $\lambda \in \sigma(T) \setminus \overline{\rho^{\pm}_{sf}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T) \cup \pi_0(T)$,
2. $\rho^{-}_{sf}(T) \cap \sigma_p(T) = \emptyset$,
3. $\rho^{-\infty}_{sf}(T) = \operatorname{int} \overline{\rho^{-\infty}_{sf}(T)}$, and
4. for every $-\infty < k \leq -1$ and for each $\lambda \in \operatorname{int} \overline{\rho^{k}_{sf}(T)} \setminus \rho^{k}_{sf}(T)$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$.

If we adapt some of these conditions, it is possible to guarantee now the ν-continuity of the approximate point spectrum on Banach spaces:

Theorem 3.8. *Let $T \in B(X)$ be such that for every $\epsilon > 0$ and $\lambda \in \sigma(T) \setminus \overline{\rho^{\pm}_{sf}(T)}$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T) \cup \pi_0(T)$. If either (1) or (2) of the following conditions holds:*

1. $\rho^{\pm}_{sf}(T) = \emptyset$;
2. $T \in \Phi(X)$, $\sigma_p(T) \cap \rho^{-}_{sf}(T) = \emptyset$ *and* $\partial\sigma(T) = \partial\sigma_{ap}(T)$,

then

$$\sigma_{ap}(T) \setminus \{0\} = [\liminf \sigma_{ap}(T_n)] \setminus \{0\} = [\limsup \sigma_{ap}(T_n)] \setminus \{0\}$$

for all $T_n \xrightarrow{\nu} T$. *Moreover, if* $0 \in acc\sigma_{ap}(T)$ *then*

$$\sigma_{ap}(T_n) \to \sigma_{ap}(T).$$

Proof. Similarly to Theorem 3.3, it follows that T satisfies Browder's theorem. From conditions either (1) or (2) we have by Theorem 3.6 that

$$[\limsup \sigma_{ap}(T_n)] \setminus \{0\} \subseteq \sigma_{ap}(T) \setminus \{0\}.$$

We show that $\sigma_{ap}(T) \setminus \{0\} \subseteq [\liminf \sigma_{ap}(T_n)] \setminus \{0\}$. Take $\lambda \in \sigma_{ap}(T)$ with $\lambda \neq 0$. Suppose that $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho^{\pm}_{sf}(T)}$. Let $\epsilon > 0$, then there exists $0 < \epsilon_1 < \epsilon$ such that $B(\lambda, \epsilon_1) \cap \overline{\rho^{\pm}_{sf}(T)} = \emptyset$ and $0 \notin B(\lambda, \epsilon_1)$. From hypothesis, we have that for every $0 < r < \epsilon_1$, the ball $B(\lambda, r)$ contains a component Ω_r of $\sigma_{sf}(T) \cup \pi_0(T)$. If $\Omega_r \subseteq \pi_0(T)$, for all $0 < r < \epsilon_1$, then $\lambda \in \overline{\pi_0(T)}$ and so by Remark 3.1 (1), $\lambda \in \liminf \pi_0(T_n)(\subseteq \liminf \sigma_{ap}(T_n))$. Now, if there exists $0 < r < \epsilon_1$ such that $\Omega_r \subseteq \sigma_{sf}(T)$ then by Proposition 3.7, Ω_r is a component of $\sigma_e(T)$. Therefore, $B(\lambda, \epsilon_1)$ is an open set such that $0 \notin B(\lambda, \epsilon_1)$ and contains a component of $\sigma(\pi(T))$. Since $T_n \xrightarrow{\nu} T$ we have that $\pi(T_n) \xrightarrow{\nu} \pi(T)$ in the Calkin algebra $B(X)/K(X)$, thus by Lemma 2.9, there exists $n_0 \in \mathbb{N}$ such that $B(\lambda, \epsilon_1)$ contains a component Ω_n of $\sigma(\pi(T_n))$ for all $n \geq n_0$. Consequently

$$\emptyset \neq \partial\Omega_n \subseteq \partial\sigma_e(T_n) \cap B(\lambda, \epsilon),$$

for all $n \geq n_0$. On the other hand, since $\sigma_e(T) = \sigma_{sf}(T) \cup \rho^{\pm\infty}_{sf}(T)$ and $\rho^{\pm\infty}_{sf}(T) \subseteq \text{int}\, \sigma_e(T)$, it follows that $\partial\sigma_e(T) = \sigma_e(T) \setminus \text{int}\, \sigma_e(T) \subseteq \sigma_e(T) \setminus \rho^{\pm\infty}_{sf}(T) \subseteq \sigma_{sf}(T)$. Therefore $\sigma_{sf}(T_n) \cap B(\lambda, \epsilon)$ for all $n \geq n_0$. This implies that

$$\lambda \in \liminf \sigma_{sf}(T_n)(\subseteq \liminf \sigma_{ap}(T_n)).$$

Suppose now that $\lambda \notin \sigma_{sf}(T) \setminus \overline{\rho^{\pm}_{sf}(T)}$, then $\lambda \in \rho_{sf}(T) \cup \overline{\rho^{\pm}_{sf}(T)}$. This implies that $i(T - \lambda) = 0$ or $\lambda \in \overline{\rho^{+}_{sf}(T)} \cup \overline{\rho^{-}_{sf}(T)}$. If $i(T - \lambda) = 0$, then since T satisfies Browder's theorem it follows that $\lambda \in \pi_0(T)$. Therefore $\lambda \in \liminf \pi_0(T_n)(\subseteq \liminf \sigma_{ap}(T_n))$. Observe that if condition (1) holds, then the proof is concluded. We assume now that (2) holds. If $\lambda \in \overline{\rho^{+}_{sf}(T)}$, then by Remark 3.1 (3), $\lambda \in \liminf \sigma_{ap}(T_n)$. Suppose that $\lambda \in \overline{\rho^{-}_{sf}(T)}$. We claim that $\lambda \in \partial\sigma_{ap}(T)$, indeed if $\lambda \in \text{int}\, \sigma_{ap}(T)$, then there exists $\epsilon_2 > 0$ such that $B(\lambda, \epsilon_2) \subseteq \sigma_{ap}(T)$. The ball $B(\lambda, \epsilon_2)$ meets $\rho^{-}_{sf}(T)$, which implies that there exists $\xi \in B(\lambda, \epsilon_2)$ such that $\xi \in \rho^{-}_{sf}(T)$. Since $\sigma_p(T) \cap \rho^{-}_{sf}(T) = \emptyset$, it follows that $T - \xi$ is injective and has closed range, hence $\xi \notin \sigma_{ap}(T)$, which is a contradiction. Therefore, $\lambda \in \partial\sigma_{ap}(T)$ and by hypothesis $\lambda \in \partial\sigma(T)$. Consequently, for every $r > 0$, the ball $B(\lambda, r)$ contains two points w_1, w_2 such that $i(T - w_1) \neq i(T - w_2)$. Thus, by Proposition 3.2, $\lambda \in \liminf \sigma_{sf}(T_n)$ $(\subseteq \liminf \sigma_{ap}(T_n))$. □

Example 3.9. *The operator A given in Example 3.4 satisfies the hypotheses of the previous theorem:* *Then* $\sigma_{ap}(A) \setminus \{0\} = [\liminf \sigma_{ap}(A_n)] \setminus \{0\} = [\limsup \sigma_{ap}(A_n)] \setminus \{0\}$, *for all* $A_n \xrightarrow{\nu} A$.

4. On a certain class of operators

We say that an operator $T \in B(H)$ is essentially G_1 if $\|(\pi(T) - z)^{-1}\| = \frac{1}{d(z, \sigma_e(T))}$, for all $z \notin \sigma_e(T)$. In [12, Theorem 6] it is shown that the restriction of the Weyl spectrum on the class of essentially G_1 operators is continuous. This is also true for ν-continuity with an additional condition for 0, as the following three theorems state.

Theorem 4.1. *Let $T \in B(H)$ and (T_n) be a sequence of essentially G_1 operators such that $T_n \xrightarrow{\nu} T$. If 0 is an accumulation point of $\sigma_e(T)$, then $\sigma_e(T_n) \to \sigma_e(T)$.*

Proof. It is an immediate consequence of Theorem 2.11. Indeed, if $\mathcal{A} = B(H)/K(H)$ the Calkin algebra, then $\pi(T_n) \xrightarrow{\nu} \pi(T)$ in $\mathcal{A}$ and $\pi(T_n) \in \mathcal{R}_{\mathcal{A}}$ for all $n \in \mathbb{N}$, thus $\sigma(\pi(T_n)) \to \sigma(\pi(T))$, i.e. $\sigma_e(T_n) \to \sigma_e(T)$. □

Corollary 4.2. *Let $T \in \Phi(H)$ be such that $0 \notin \sigma_w(T)$ or $0 \in acc\sigma_w(T)$. If (T_n) is a sequence of essentially G_1 operators such that $T_n \xrightarrow{\nu} T$ then $\sigma_w(T_n) \to \sigma_w(T)$.*

Proof. From [3, Theorem 3.3] we have that $\limsup \sigma_w(T_n) \subseteq \sigma_w(T)$. Let $\lambda \in \sigma_w(T) \setminus \{0\}$. If $\lambda \in \sigma_e(T)$ then $\lambda \in \liminf \sigma_e(T_n) \subseteq \liminf \sigma_w(T_n)$. If $\lambda \notin \sigma_e(T)$ then $\lambda \in \rho^{\pm}_{sf}(T)$, therefore by Remark 3.1 (5), $\lambda \in \liminf \sigma_w(T_n)$. □

Corollary 4.3. *Let $T \in \Phi(H)$ be such that satisfies Browder's theorem. If $0 \in acc\sigma_w(T)$ and (T_n) is a sequence of essentially G_1 operators such that $T_n \xrightarrow{\nu} T$, then $\sigma(T_n) \to \sigma(T)$.*

Proof. Let $\lambda \in \sigma(T)$. If $\lambda \in \sigma_w(T)$ then, by Corollary 4.2, $\lambda \in \liminf \sigma_w(T) \subseteq \liminf \sigma(T)$. If $\lambda \notin \sigma_w(T)$ then since T satisfies Browder's theorem it follows that $\lambda \in \pi_0(T)$. Therefore, by Remark 3.1 (1), $\lambda \in \liminf \pi_0(T_n) \subseteq \liminf \sigma(T_n)$. □

Let $0 < p \leq 1$, an operator $A \in B(H)$ is called p-hyponormal, if $(A^*A)^p - (AA^*)^p \geq 0$. For the case $p = 1$ the operator A is called hyponormal. In [9] and [11], Djordjevic et al. and independently Hwang et al. prove that the restriction of the spectrum on the class of p-hyponormal operators is continuous. The proof of the following theorem is an adaptation of the idea of [9] for the case of ν-convergence. Note first that if $0 < p < \frac{1}{2}$ and $A \in B(H)$ is a p-hyponormal operator, then by [6], $N(A) \subseteq N(A^*)$, which implies that $N(A)$ is invariant for both A and A^*. Therefore

$$A = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \tag{9}$$

on $N(A) \oplus N(A)^{\perp}$, where $B = A|_{N(A)^{\perp}}$ and $0 \notin \sigma_p(B)$. From [19, Lemma 4], B is also p-hyponormal. We claim that $0 \notin \sigma(|B|)$, indeed if $0 \in \sigma(|B|)(= \sigma_{ap}(|B|))$ then there exists a sequence (x_m) of unit vectors such that $|B|x_m \to 0$. This implies that $Bx_m \to 0$, thus $0 \in \sigma_{ap}(B)$, but since $R(A)$ is closed i.e. $R(B)$ is closed, it follows that $0 \in \sigma_p(B)$, which is a contradiction.

Theorem 4.4. *Let $0 < p < 1/2$. If T_n, T are operators in $B(H)$ such that*

1. $T_n \xrightarrow{\nu} T$,
2. *$T \in \Phi(H)$ and T_n is p-hyponormal for all $n \in \mathbb{N}$,*
3. *the sequence $(\||B_n|^{-1}\|)$ is bounded, where the operators B_n are as in (9),*

then $\sigma(T_n) \to \sigma(T)$.

Proof. First observe that $\|T_nT - T^2\| = \|(T_n - T)T\| \to 0$, thus $T_nT \xrightarrow{n} T^2$, which implies that $T_nT \in \Phi(H)$, for all n large. Thus we may suppose that $R(T_n)$ is closed for all $n \in \mathbb{N}$. We show that there exist a sequence (S_n) of hyponormal operators and a sequence (X_n) of invertible operators such that $S_n = X_nT_nX_n^{-1}$, for all $n \in \mathbb{N}$, and $(\|X_n\|)$, $(\|X_n^{-1}\|)$ are bounded.

From (9) we have that $T_n = \begin{bmatrix} 0 & 0 \\ 0 & B_n \end{bmatrix}$ on $N(T_n) \oplus N(T_n)^{\perp}$, $0 \notin \sigma_p(B_n)$, B_n is p-hyponormal and $0 \notin \sigma(|B_n|)$. Consider the polar decomposition $B_n = U_n|B_n|$ and define $\widehat{B_n} = |B_n|^{1/2}U_n|B_n|^{1/2}$. From [2, Theorem 2], $\widehat{B_n}$ is $(p + \frac{1}{2})$-hyponormal. Observe that if $x \in N(\widehat{B_n})$ then $|B_n|^{1/2}U_n|B_n|^{1/2}x = 0$ and so $B_n|B_n|^{-1}|B_n|^{1/2}x = 0$ which implies that $|B_n|^{-1}|B_n|^{1/2}x = 0$ because $0 \notin \sigma_p(B_m)$, hence $x = 0$. Thus $N(\widehat{B_n}) = \{0\}$ i.e. $0 \notin \sigma_p(\widehat{B_n})$. This implies that $0 \notin \sigma(|\widehat{B}_n|)$. Let $\widehat{B}_n$ have the polar decomposition $\widehat{B}_n = V_n|\widehat{B}_n|$, then by [2, Theorem 1], the operator $\widetilde{B}_n$, defined by $\widetilde{B}_n = |\widehat{B}_n|^{1/2}V_n|\widehat{B}_n|^{1/2}$, is hyponormal. Define

$$X_n = \begin{bmatrix} 1 & 0 \\ 0 & |\widehat{B}_n|^{1/2}|B_n|^{1/2} \end{bmatrix} \quad \text{and} \quad S_n = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{B}_n \end{bmatrix}.$$

Then S_n is hyponormal, X_n is invertible and $S_n = X_n T_n X_n^{-1}$. From condition (3) we have that $(\|X_n^{-1}\|)$ is bounded. Also it is clear that $(\|X_n\|)$ is bounded.

We show that $\sigma(T) \subseteq \liminf \sigma(T_n)$. Take $\lambda \in \sigma(T) \setminus \{0\}$ and suppose that $\lambda \notin \liminf \sigma(T_n)$. Then we may assume that there exists $\epsilon > 0$ such that $B(\lambda, \epsilon) \cap \sigma(T_n) = \emptyset$ for all $n \in \mathbb{N}$. This implies that $T_n - \lambda$ is invertible. In a similar way to proof of Theorem 4.1, we have that

$$\left\|\left[\left(T_n - T)(T_n - \lambda\right)^{-1}\right]^2\right\| \leq \frac{1}{|\lambda|}\Big[\|(T_n - T)T_n\|\|(T_n - \lambda)^{-1}\|\|T_n - T\| + \|(T_n - T)T\| + \|(T_n - T)T_n\|\Big]\|(T_n - \lambda)^{-1}\|. \tag{10}$$

Since T_n and S_n are similar, it follows that $\sigma(T_n) = \sigma(S_n)$. Therefore, $d(\lambda, \sigma(S_n)) \geq \epsilon$ and $S_n - \lambda$ is invertible. Note that $(T_n - \lambda)^{-1} = X_n^{-1}(S_n - \lambda)^{-1}X_n$. Moreover, since S_n is hyponormal it follows that $\|(S_n - \lambda)^{-1}\| = \frac{1}{d(\lambda, \sigma(S_n))} \leq \frac{1}{\epsilon}$. Thus the right term of (10) is bounded by

$$\frac{1}{|\lambda|}\Big[\|(T_n - T)T_n\|\frac{M_1M_2}{\epsilon}\|T_n - T\| + \|(T_n - T)T\| + \|(T_n - T)T_n\|\Big]\frac{M_1M_2}{\epsilon}, \tag{11}$$

where M_1, M_2 are constants such that $\|X_n\| \leq M_1$ and $\|X_n^{-1}| \leq M_2$, for all $n \in \mathbb{N}$. Since $T_n \xrightarrow{\nu} T$ it follows that the expression in (11) tends to zero. Proceeding similarly to the final part of the proof of Theorem 4.1, we obtain that $T - \lambda$ is invertible, which is a contradiction. □

Remark 4.5. *The conclusion of Theorem 4.4 holds if we replace the hypothesis by the following conditions:*

1. $T_n \xrightarrow{\nu} T$, $T^*(T_n - T) \xrightarrow{n} 0$ *and* $T_n^*(T_n - T) \xrightarrow{n} 0$;
2. $T \in \Phi(H)$ *and* $\{0\} \neq N(T) \subseteq N(T_n)$ *for all* $n \in \mathbb{N}$;
3. T, T_n *are p-hyponormal operators.*

Indeed, from condition (1), $|T_n|^2 - |T|^2 = T_n^*T_n - T^*T = T_n^*(T_n - T) + [T^*(T_n - T)]^* \xrightarrow{n} 0$, thus $|T_n|^{1/2} \xrightarrow{n} |T|^{1/2}$. Since $0 \in \sigma_p(T)$, it follows that $T = 0 \oplus B$ on $N(T) \oplus N(T)^\perp$ with $0 \notin \sigma(|B|)$. Then there exists $\alpha > 0$ such that $\alpha\|y\| \leq \||B|^{1/2}y\|$ for all $y \in N(T)^\perp$. This implies by condition (2) that for $0 < \epsilon < \alpha$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $(\alpha - \epsilon)\|y\| \leq \||B_n|^{1/2}y\|$ for all $y \in N(T_n)^\perp$. Consequently, $\|(|B_n|^{1/2})^{-1}\| \leq \frac{1}{\alpha - \epsilon}$, for all $n \geq N$.

Berberian in [4] shows that for every Hilbert space H, there exists a Hilbert space $K \supset H$ and a faithful $*-$representation $T \to T^\circ$ from $B(H)$ to $B(K)$: $(S + T)^\circ = S^\circ + T^\circ$, $(\lambda T)^\circ = \lambda T^\circ$, $(ST)^\circ = S^\circ T^\circ$, $(T^*)^\circ = (T^\circ)^*$, $(I_H)^\circ = I_K$ and $\|T^\circ\| = \|T\|$ such that

1. $T \geq 0$ if and only if $T^\circ \geq 0$,
2. $\sigma_p(T^\circ) = \sigma_{ap}(T^\circ) = \sigma_{ap}(T)$.

Remark 4.6. *Observe that in the previous theorem,* $\sigma_p(B_n) = \sigma_{ap}(B_n)$ *due to* $R(T_n)$ *is closed. In [9] the authors use the Berberian extension* T_n° *of a p-hyponormal operator* T_n *and state that if* $0 \in \sigma_p(T_n^\circ)$*, then*

$$\sigma_p(B_n) = \sigma_{ap}(B_n), \tag{12}$$

where $T_n^\circ = 0 \oplus B_n$ *on* $N(T_n) \oplus N(T_n)^\perp$ *and* $0 \notin \sigma_p(B_n)$*, without the need for* $R(T_n^\circ)$ *to be closed. This fact was also established in [10], page 586, line 20. The authors claim that*

$$\sigma_{ap}(B_\lambda) = \sigma_p(B_\lambda), \tag{13}$$

for all non-zero $\lambda \in \sigma_p(A^\circ)$, where $A \in \mathcal{C}(i)$, and this collection is defined as the set of all operators $T \in B(H_i)$ for which $\sigma(T) = \{0\}$ implies T is nilpotent and T° satisfies the property:

$$T^\circ = \begin{bmatrix} \lambda & X_\lambda \\ 0 & B_\lambda \end{bmatrix} \quad on \quad N(T^\circ - \lambda) \oplus N(T^\circ - \lambda)^\perp$$

at every non-zero $\lambda \in \sigma_p(T^\circ)$ for some operators X_λ and B_λ such that $\lambda \notin \sigma_p(B_\lambda)$ and $\sigma(T^\circ) = \{\lambda\} \cup \sigma(B_\lambda)$. However, equalities (12) and (13) are not necessary hold. We prove only that (13) is false: It is clear that $\sigma_{ap}(B_\lambda) \setminus \{\lambda\} = \sigma_p(B_\lambda)$ and $\alpha(B_\lambda - \lambda) = 0$, but $R(B_\lambda - \lambda)$ is not necessarily closed. Indeed, consider a normal operator $A \in B(H_i)$ such that $\sigma(A) = [0, 1]$ (for example, the multiplication operator $A : L^2([0, 1]) \to L^2([0, 1])$ defined by $A(f)(x) = xf(x)$). Then $A \in \mathcal{C}(i)$. We show that for every $\lambda \in \sigma_p(A^\circ)$, $R(B_\lambda - \lambda)$ is not closed. By contradiction, suppose that there exists $\lambda \in \sigma_p(A^\circ)$ such that $R(B_\lambda - \lambda)$ is closed. Then $B_\lambda - \lambda$ is a semi-Fredholm operator such that $\alpha(B_\lambda - \lambda) = 0$. By [5, Theorem 4.2.1], there exists $\epsilon > 0$ such that if $|\gamma - \lambda| < \epsilon$ then $B_\lambda - \gamma \in \Phi_+(N(A^\circ - \lambda)^\perp)$ and $\alpha(B_\lambda - \gamma) = 0$. This implies that $R(A^\circ - \gamma) = (\lambda - \gamma)N(A^\circ - \lambda) \oplus R(B_\lambda - \gamma)$ is closed and $\alpha(A^\circ - \gamma) = \alpha((\lambda - \gamma)I) + \alpha(B_\lambda - \gamma) = 0$ for all $\gamma \in B(\lambda, \epsilon)$ with $\gamma \neq \lambda$. Therefore, $\lambda \in iso\,\sigma_{ap}(A^\circ)$. On the other hand, since A is a normal operator it follows that $\sigma(A) = \sigma_{ap}(A) = \sigma_{ap}(A^\circ)$. Thus, $\lambda \in iso\,\sigma(A)$, which is a contradiction, because $\sigma(A) = [0, 1]$. Consequently, the equality (13) is not true. This affects the proof of the main result in the paper [10].

References

[1] Mario Ahues, Alain Largillier, and Balmohan V. Limaye, *Spectral computations for bounded operators*, Appl. Math. (Boca Raton), vol. 18, Boca Raton, FL: Chapman & Hall/CRC, 2001 (English).

[2] Ariyadasa Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Oper. Theory **13** (1990), no. 3, 307–315 (English).

[3] Aymen Ammar, *Some properties of the Wolf and Weyl essential spectra of a sequence of linear operators ν-convergent*, Indag. Math., New Ser. **28** (2017), no. 2, 424–435 (English).

[4] S. K. Berberian, *Approximate proper vectors*, Proc. Am. Math. Soc. **13** (1962), 111–114 (English).

[5] S. R. Caradus, W. E. Pfaffenberger, and Bertram Yood, *Calkin algebras and algebras of operators on Banach spaces*, Lect. Notes Pure Appl. Math., vol. 9, CRC Press, Boca Raton, FL, 1974 (English).

[6] Muneo Chō and Tadasi Huruya, *p-hyponormal operators for* $0 < p < \frac{1}{2}$, Ann. Soc. Math. Pol., Ser. I, Commentat. Math. **33** (1993), 23–29 (English).

[7] John B. Conway and Bernard B. Morrel, *Operators that are points of spectral continuity*, Integral Equations Oper. Theory **2** (1979), 174–198 (English).

[8] John B. Conway and Bernard B. Morrel, *Operators that are points of spectral continuity. II*, Integral Equations Oper. Theory **4** (1981), 459–503 (English).

[9] S. V. Djordjević and B. P. Duggal, *Weyl's theorems and continuity of spectra in the class of p-hyponormal operators*, Stud. Math. **143** (2000), no. 1, 23–32 (English).

[10] B. P. Duggal, I. H. Jeon, and I. H. Kim, *Continuity of the spectrum on a class of upper triangular operator matrices*, J. Math. Anal. Appl. **370** (2010), no. 2, 584–587 (English).

[11] In Sung Hwang and Woo Young Lee, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), no. 1, 151–157 (English).

[12] Glenn R. Luecke, *A note on spectral continuity and on spectral properties of essentially* G_1 *operators*, Pac. J. Math. **69** (1976), 141–149 (English).

[13] Vladimir Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, 2nd ed. ed., Oper. Theory: Adv. Appl., vol. 139, Basel: Birkhäuser, 2007 (English).

[14] Gerard J. Murphy, *C*-algebras and operator theory*, Boston, MA etc.: Academic Press, Inc., 1990 (English).

[15] J. D. Newburgh, *The variation of spectra*, Duke Math. J. **18** (1951), 165–176 (English).

[16] S. Sánchez-Perales, S. V. Djordjević, and S. Palafox, *Some results about spectral continuity and compact perturbations*, Filomat **34** (2020), no. 14, 4837–4845 (English).

[17] Salvador Sánchez-Perales and Slaviša V. Djordjević, *Spectral continuity using ν-convergence*, J. Math. Anal. Appl. **433** (2016), no. 1, 405–415 (English).

[18] Martin Schechter, *Principles of functional analysis.*, 2nd ed. ed., Grad. Stud. Math., vol. 36, Providence, RI: American Mathematical Society (AMS), 2001 (English).

[19] Atsushi Uchiyama, *Berger-Shaw's theorem for p-hyponormal operators*, Integral Equations Oper. Theory **33** (1999), no. 2, 221–230 (English).