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# Fixed point theorems and applications in fuzzy quasi-metric spaces

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**Abstract.** This paper delves into the realm of fuzzy quasi-metric spaces, emphasizing the identification of fixed points for specific mappings within this unique framework. We have formulated fixed-point theorems designed for various mapping types in these spaces, underscoring their practical relevance and applicability. Moreover, we have uncovered principles akin to Kransnoselski's theorems, specifically tailored to fuzzy quasi-metric spaces. Our research further investigates the stability of fixed points under certain conditions, providing a comprehensive analysis of their behavior. To elucidate our findings, we have incorporated illustrative examples, which serve to clarify and exemplify the theoretical concepts discussed. This inclusion of examples not only enhances understanding but also demonstrates the practical implementation of our theorems. Through our work, we aim to contribute significantly to the recent advancements in the study of fuzzy quasi-metric spaces and fixed-point theory. Our findings offer valuable insights and extend the current knowledge base, paving the way for future research and applications in this evolving field. Ultimately, our paper stands as a testament to the ongoing progress and innovation in fuzzy quasi-metric space theory, reflecting its growing importance and potential for further exploration.

## 1. Introduction

A contraction mapping, also known as an antithesis mapping, is a fundamental concept in the context of fuzzy quasi-metric spaces. This type of mapping is crucial for establishing fixed-point theorems and understanding the properties of these spaces. In a fuzzy quasi-metric space, a contraction mapping serves to demonstrate the existence and properties of points that contradict certain conditions or relationships within the space (see [13, 17]).

Essentially, a contraction mapping identifies points where the behavior or properties of elements in the space seem to oppose or contradict what one might expect based on the space's structure or defined relationships. These mappings are particularly useful in dealing with nonlinear and complex systems where conventional metric spaces may not apply due to their stricter properties (see [14]).

The study of contraction mappings is vital in various mathematical and scientific fields, including nonlinear analysis, optimization, control theory, and mathematical modeling. By utilizing contraction mappings, researchers can gain insights into the existence and behavior of solutions, fixed points, and critical points in fuzzy quasi-metric spaces, thus contributing to the understanding of complex systems and providing valuable tools for solving practical problems (see [15]).

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In summary, a contraction mapping in a fuzzy quasi-metric space helps identify and study points that contradict expected behaviors or relationships within the space, making it a valuable concept in various areas of mathematics and science.

In their work, Kramosil and Michalek introduced and explored the concept of a fuzzy metric space, significantly connecting it to a category of probabilistic metric spaces known as (generalized) Menger spaces (see [11]). Subsequently, George and Veeramani undertook a more robust examination of metric fuzziness (see [5], with additional references in [7]). It is commonly recognized that each metric gives rise to a fuzzy metric according to George and Veeramani's perspective. Conversely, every fuzzy metric space as defined by George and Veeramani, as well as Kramosil and Michalek, gives rise to a topology that can be metrized (see [6] and [9]).

On the other hand, it is widely recognized that quasi-metric spaces provide a valuable framework for addressing and resolving numerous challenges in topological algebra, approximation theory, theoretical computer science, and more, as detailed in [12, 19].

In their publication [9], two distinct concepts of fuzzy quasi-metric spaces are introduced, extending the analogous notions found in the works of Kramosil and Michalek and George and Veeramani to the realm of quasi-metrics. The authors establish several fundamental properties of these spaces, demonstrating that each quasi-metric naturally gives rise to a fuzzy quasi-metric, and conversely, every fuzzy quasi-metric leads to the creation of a quasi-metrizable topology. These findings provide a valuable foundation for deriving numerous characteristics and properties of fuzzy quasi-metric spaces.

In a recent publication by V. Gregori et al.[4], a clear and precise notion of completeness is introduced. Additionally, they propose a method for constructing the completion of a specific class of $T_2$-fuzzy quasi-metric spaces. This construction is influenced by similar principles originally introduced by Doitchinov, as detailed in [2].

In a recent development, Romaguera [16] introduced comprehensive fixed-point theorems applicable to both left and right complete Hausdorff KM-fuzzy quasi-metric spaces. These theorems have been utilized to derive characterizations of these specific forms of fuzzy quasi-metric completeness.

Recently, S. U. Rehman and colleagues introduced a groundbreaking concept in their work referenced as [13]. This novel concept involves rational type fuzzy-contraction mappings within the framework of fuzzy metric spaces. The authors established a series of fixed-point results by applying the rational type fuzzy-contraction conditions and provided illustrative examples to substantiate their findings. This innovative concept holds significant promise within the realm of fuzzy fixed-point theory and exhibits potential for generalization to various types of contractive mappings in fuzzy metric spaces, as also indicated in [14, 15].

The significance of this study lies in its exploration of fixed-point theorems within fuzzy quasi-metric spaces, offering new theoretical insights and practical applications. By developing theorems for various mappings and identifying conditions for the stability of fixed points, the research advances the understanding of fuzzy quasi-metric spaces. The inclusion of examples not only clarifies the concepts but also demonstrates the practical relevance of the theorems. This study contributes to the broader field of fuzzy quasi-metric spaces, fostering further innovation and exploration, while providing a solid foundation for future research and potential applications in related mathematical fields.

In this research, we explore fuzzy quasi-metric spaces and find fixed points for certain mappings in this context. We develop fixed-point theorems for several kinds of mappings, emphasizing their usefulness and importance in practice. Furthermore, we explore the stability of fixed points under specific conditions and reveal concepts akin to Krasnoselskii's theorems, modified for fuzzy quasi-metric spaces. To elucidate and illustrate the theoretical notions, examples are provided. By providing insightful information and expanding the body of knowledge, our work greatly advances fuzzy quasi-metric spaces and fixed-point theory, opening the door for further study and applications.

The structure of the paper is as follows, we begin in the next part by discussing and thoroughly examining the fundamental concepts and preliminary results of fuzzy quasi-metric (quasi-normed) spaces. Theorems about fixed points that apply to various kinds of mappings that contract in the fuzzy quasi-metric space are covered in section 3. Examining the fixed point theorem in relation to fuzzy quasi-metric spaces is the focus of Section 4. A number of fixed point theorems that bear similarities to Kransnoselski's contributions in the field of FM-spaces are covered in the concluding section. We also study the effect of parameter changes on continuity at fixed places.

## 2. Preliminaries And Notations

Let's revisit the concept of a quasi-metric defined on a set $X$. A quasi-metric on $X$ is a real-valued function denoted as $d$ and defined on $X \times X$, where the following conditions hold for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

In accordance with contemporary terminology, as elucidated in Section 11 of [12], when we refer to a quasi-metric on $X$, we imply a quasi-metric $d$ on $X$ that satisfies the specific condition that $d(x, y) = d(y, x) = 0$ if and only if $x = y$. If the quasi-metric $d$ adheres to the even stricter condition that $d(x, y) = 0$ if and only if $x = y$, we classify it as a $T_1$ quasi-metric on $X$.

Let's remember, as mentioned in reference [18], that a continuous $t$-norm can be defined as a binary operation denoted as $\gamma : [0, 1] \times [0, 1] \to [0, 1]$. This operation should satisfy the condition that the set $([0, 1], \leqslant, \gamma)$ forms an ordered Abelian topological monoid with unit 1.

**Definition 2.1.** *A fuzzy quasi-metric space can be defined as a structured trio denoted as $(X, M, \gamma)$, where $X$ represents a nonempty set, $\gamma$ is a continuous t-norm, and $M$ is a fuzzy set defined over $X \times X \times (0, 1)$. This definition is subject to the following conditions, which hold true for all elements $x, y, z \in X$, and positive values s and t:*

*(FQM1)* $M(x, y, t) > 0$;
*(FQM2)* $M(x, y, t) = 1$ *if and only if* $x = y$;
*(FQM3)* $M(x, z, t + s) \geqslant \gamma(M(x, y, t), M(y, z, s))$;
*(FQM4)* $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous.*

Condition (FQM2) is equivalent to the following:

$$M(x, x, t) = 1 \text{ for all } x \in X \text{ and } t > 0, \text{ and } M(x, y, t) < 1 \text{ for all } x \neq y \text{ and } t > 0,$$

We can call $(M, \gamma)$ a fuzzy quasi-metric on the set $X$ when we have a fuzzy quasi-metric space represented as $(X, M, \gamma)$. Sometimes this may be simplified to say that $(X, M)$ is a fuzzy quasi-metric space, or just that $M$ is a fuzzy quasi-metric, where there is no possibility of misunderstanding. According to George and Veeramani's definition [5], a fuzzy quasi-metric $M$ is categorized as a fuzzy metric if it meets the requirement that $M(x, y, t) = M(y, x, t)$ for all values of $t > 0$.

We will often rely on the following two widely recognized facts without explicitly stating them:

(a) $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.
(b) If $r > s$ we can find $t$ such that $\gamma(r, t) > s$, where $r, s, t \in (0, 1)$.

**Example 2.2.** *Consider the quasi-metric space $(X, d)$ as discussed in [3]. Let's define a function $\gamma(a, b)$ for all a and b within the closed interval $[0, 1]$ as $\gamma(a, b) = ab$. Additionally, we can define the function $M_d$ on the product space $X \times X \times (0, \infty)$ as follows:*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

*Now, $(X, M_d)$ constitutes a fuzzy quasi-metric space, and the operation $(M_d, \cdot)$ is commonly referred to as the (standard) fuzzy quasi-metric that arises from the quasi-metric d, as detailed in references [5] and [9].*

As mentioned in [17], other examples of fuzzy quasi-metrics can be investigated by exploring fuzzy metrics. Naturally, a $T_1$-topology, represented by $\tau_M$, arises for each given fuzzy quasi-metric $M$ defined on the set $X$. A basis of open sets of the type $\{B_M(x, r, t) : x \in X, r \in (0, 1), t > 0\}$ is used to create this topology, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $r \in (0, 1)$.

When we have a quasi-metric space $(X, d)$, the topology induced by the quasi-metric $d$ coincides with the topology denoted as $\tau_{M_d}$ generated by the fuzzy quasi-metric $(M_d, \gamma)$.

A topological space $(X, \tau)$ is said to admit a compatible fuzzy quasi-metric if there exists a fuzzy quasi-metric $M$ defined on $X$ such that $\tau = \tau_M$. The previous result implies that every topological space that can be endowed with a quasi-metric topology also permits a compatible fuzzy quasi-metric. Conversely, for a given fuzzy quasi-metric space $(X, M)$, the family $\{U_n : n \in \mathbb{N}\}$, where $U_n = \left\{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\right\}$, constitutes a countable basis for a quasi-uniformity $U_M$ on $X$ that aligns with $\tau_M$, as stated in [9]. Consequently, the topological space $(X, \tau_M)$ is amenable to a quasi-metric structure, as detailed in [3].

**Definition 2.3.** *[1] A fuzzy quasi-norm on a real linear space X is a pair $(N, \gamma)$ such that $\gamma$ is a continuous t-norm and N is a fuzzy set in $X \times (0, \infty)$ satisfying the following conditions: for any $x, y \in X$,*

*(QFN1)* $N(x, 0) = 0$;

*(QFN2)* $N(x, t) = N(-x, t) = 1$, *for all* $t > 0$ *if and only if* $x = \theta$;

*(QFN3)* $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right)$, *for all* $\lambda, t > 0$;

*(QFN4)* $N(x + y, t + s) \geqslant \gamma(N(x, t), N(y, s))$, *for all* $t, s > 0$;

*(QFN5)* $N(x, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous;*

*(QFN6)* $\lim_{t\to\infty} N(x, t) = 1$.

"Evidently, the function $N(x, \cdot)$ exhibits monotonicity for every $x$ in the set $X$." "A fuzzy quasi-norm $(N, \gamma)$ earns the label 'fuzzy norm' if it satisfies the property:$N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ for all $x$ in $X$ and all $c$ in the real numbers except zero." "Let $(N, \gamma)$ be a fuzzy quasi-norm, define the following:

$$N^{-1}(x, t) = N(-x, t) \quad \text{and } N^s(x, t) = \min\{N(x, t), N^{-1}(x, t)\}.$$

These definitions apply to all $x$ in $X$ and $t$ greater than zero. Then, $N^{-1}(x, t)$ becomes a fuzzy quasi-norm, and $N^s(x, t)$ becomes a fuzzy norm on the space $X$." "In case $(N, \gamma)$ serves as a fuzzy quasi-norm (or fuzzy norm) on $X$, we dub the triple $(X, N, \gamma)$ as a fuzzy quasi-normed space (or a fuzzy normed space)."

Every fuzzy quasi-norm $(N, \gamma)$ defined on the set $X$ induces a topology denoted as $\tau_N$. This topology possesses a foundational set consisting of open balls centered at each point $x$ in $X$, defined as follows:

$$\mathcal{B}(x) = \{B_N(x, r, t) : r \in (0, 1), t > 0\},$$

where

$$B_N(x, r, t) = \{y \in X : N(y - x, t) > 1 - r\}.$$

It's evident that the topology $\tau_N$ is $T_0$ and satisfies the first countability axiom. Additionally, due to the property $x + B_N(\theta, r, t) = B_N(x, r, t)$, the topology $\tau_N$ exhibits translational invariance. In terms of convergence in this topology, a sequence $x_n$ in $X$ converges to the point $x$ with respect to $\tau_N$ (denoted as $x_n \xrightarrow{\tau_N} x$) if and only if $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.

## 3. Fixed point theorems for Contraction mapping in fuzzy quasi-metric space

Here we introduce fixed point theorems for several contraction type mappings in the framework of a fuzzy quasi-metric space.

**Definition 3.1.** *Consider a fuzzy quasi-metric space denoted as $(X, M)$. A mapping $T : X \to X$ qualifies as a contraction mapping in this context if, and only if, there exists a value $\alpha \in (0, 1)$ satisfying the following condition for all $\xi, \varsigma \in X$ and $t > 0$:*

$$M(T\xi, T\varsigma) \geqslant M\left(\xi, \varsigma, \frac{t}{\alpha}\right) \tag{1}$$

*This condition characterizes the property of T being a contraction mapping on $(X, M)$.*

**Lemma 3.2.** *Consider a fuzzy quasi-metric space denoted as $(X, M, \gamma)$, where $\gamma$ is a continuous t-norm. Suppose we have a mapping $T : X \to X$ that is a contraction mapping, satisfying condition (1). In such a case, one of the following two situations holds: (i) The mapping T possesses a unique fixed point. Or (ii) For any initial point $\xi_0 \in X$, the supremum of the set $\{G_{\xi_0}(t) : t \in \mathbb{R}\}$ is less than 1, where $G_{\xi_0} = \inf\{M(\xi_0, \xi_m, t) : \xi_m = T\xi_{m-1}, m \in \mathbb{N}\}$.*

*Proof.* Suppose there exists a point $\xi_0 \in X$ such that $\sup\{G_{\xi_0}(t) : t \in \mathbb{R}\} = 1$. Under this condition, we can establish the following inequality:

$$M(\xi_n, \xi_{n+m}, t) \geqslant M\left(\xi_0, \xi_m, \frac{t}{\alpha^n}\right).$$

Since $G_{\xi_0}$ is non-decreasing, it follows that:

$$\lim_{n\to\infty} M(\xi_n, \xi_{n+m}, t) = 1$$

for all $t > 0$, independent of $m$. In other words, the sequence $\xi_n$ is a Cauchy sequence within the complete fuzzy quasi-metric space $(X, M)$. Consequently, there exists a point $\xi^* \in X$ to which the sequence $\xi_n$ converges. To demonstrate that $T\xi^* = \xi^*$, we can observe that for every $n \in \mathbb{N}$:

$$\begin{aligned} M(T\xi^*, \xi^*, t) &\geqslant \gamma\left(M\left(T\xi^*, \xi_n, \frac{t}{2}\right), M\left(\xi_n, \xi^*, \frac{t}{2}\right)\right) \\ &\geqslant \gamma\left(M\left(\xi^*, \xi_{n-1}, \frac{t}{2}\right), M\left(\xi_n, \xi^*, \frac{t}{2}\right)\right). \end{aligned}$$

Therefore, for all $t > 0$, we have:

$$\begin{aligned} M(Txi^*, \xi^*, t) &\geqslant \lim_{n\to\infty} \gamma\left(M\left(\xi^*, \xi_{n-1}, \frac{t}{2}\right), M\left(\xi_n, \xi^*, \frac{t}{2}\right)\right) \\ &\geqslant 1. \end{aligned}$$

Hence, $\xi^*$ serves as the unique fixed point of $T$, and the proof is concluded. ■

The proof of the following result is straightforward, and therefore, we will not provide it here.

**Theorem 3.3.** *Consider a complete fuzzy quasi-metric space denoted as $(X, M, \gamma)$, and let $T : X \to X$ be a contraction mapping that satisfies condition (1). In this context, it can be concluded that T possesses a fixed point within the space X.*

As a reminder, a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is considered to satisfy condition $(\Phi)$ if it meets the following criteria: it is strictly increasing, $\psi(0) = 0$, and $\lim_{n\to\infty} \psi^n(t) = \infty$ for all $t > 0$, where $\psi^n(t)$ represents the $n$-th iteration of the function $\psi(t)$.

**Theorem 3.4.** *Let's consider a fuzzy quasi-metric space represented as $(X, M, \gamma)$, where $\gamma$ is a continuous t-norm that satisfies $\sup_{t\in[0,1]} \gamma(t, t) = 1$. Within this context, we have a mapping denoted as $T : X \to X$, which is a contraction mapping and adheres to the following conditions:*

*(a) For each $x \in X$, the supremum over $t \in \mathbb{R}$ of $G_x(t)$ is equal to 1, where*

$$G_x(t) = \inf_{n\in\mathbb{N}} M(x, T^n x, t).$$

*(b) For each $x \in X$, there exists $m(x) \in \mathbb{N}$ such that for any $y \in X$ and $t \in \mathbb{R}^+$, the inequality (2) holds, where $\Phi(t)$ is a left-continuous function satisfying condition $(\Phi)$.*

$$\begin{aligned} M(T^{m(x)}x, T^{m(x)}y, t) \geqslant\ & \gamma(M(x, y, \psi(t)), M(x, T^{m(x)}x, \psi(t)), \\ & M(x, T^{m(x)}y, \psi(t)), \end{aligned} \tag{2}$$

*Under these conditions, the following conclusions can be drawn:*

*(i) For any initial point $\xi_0 \in X$, the sequence $\xi_n$ defined by $\xi_n = T^{m(\xi_{n-1})}\xi_{n-1}$ for $n \in \mathbb{N}$ converges to some $\xi^* \in X$.*

*(ii) Furthermore, if there exists $t^* \in \mathbb{R}^+$ such that $M(\xi^*, T^{m(\xi^*)}\xi^*, t^*) = 1$, then $\xi^*$ serves as the unique fixed point of $T$ within $X$, and the iterative sequence $T^n\xi_0$ converges to $\xi^*$.*

*Proof.* (i) We establish the fact that the sequence $\xi_n$ forms a Cauchy sequence within the set $X$. For any given $n \in \mathbb{N}$, if we denote $m(\xi_i)$ as $m_i$ for $i \in \mathbb{N}$, then, as implied by equation (2), we can conclude that:

$$\begin{aligned} M(\xi_i, T^n\xi_i, t) &\geqslant \gamma\{M(\xi_{i-1}, T^n\xi_{i-1}, \psi(t)), M(\xi_{i-1}, T^{m_{i-1}+n}\xi_{i-1}, \psi(t)) \\ &\quad M(\xi_{i-1}, T^{m_{i-1}+n}\xi_{i-1}, \psi(t))\} \\ &\geqslant G_{\xi_{i-1}}(\psi(t)). \end{aligned}$$

Hence, we find that $G_{\xi_i}(t) \geqslant G_{\xi_{i-1}}(\psi(t)) \geqslant \cdots \geqslant G_{\xi_0}(\psi(t))$. Consequently, we obtain:

$$\begin{aligned} M(\xi_i, \xi_{i+j}, t) &= M(\xi_i, T^{m^{i+j-1}+\cdots+m_i}\xi_i, t) \\ &\geqslant G_{\xi_i}(t) \geqslant G_{\xi_0}(\psi^i(t)). \end{aligned} \tag{3}$$

Given that the distribution function is non-decreasing, $\sup_{t\in\mathbb{R}} G_{\xi_0}(t) = 1$, and $\psi^i(t) \to \infty$ as $i \to \infty$ for all $t > 0$, we can conclude that $G_{\xi_0}(\psi^i(t)) \to 1$ as $i \to \infty$. This implies that the sequence $\xi_n$ is a Cauchy sequence in $X$. Since $X$ is a complete space, it follows that $\xi_n$ converges to $\xi^* \in X$.
(ii) We begin by establishing that $\xi^*$ serves as a fixed point of $T^{m(\xi^*)}$. Let's denote $m(\xi^*)$ as $m^*$. Based on the assumption, there exists $t^* \in \mathbb{R}$ such that $M(\xi^*, T^{m^*}\xi^*, t) = 1$. We can represent this as follows:

$$t_0 = \inf\left\{t : M(\xi^*, T^{m^*}\xi^*, t) = 1\right\}. \tag{4}$$

It is evident that $t_0 \leqslant t^*$. Next, we aim to demonstrate that $t_0 = 0$. Indeed, if $t_0 > 0$, considering the left continuity of $\psi$, we can find $t_1$ and $t_2$ in the positive real numbers with $0 < t_2 < t_1 < t_0$ such that $\psi(t_2) > t_0$. Subsequently, from equation (4), we deduce the following:

$$M(\xi^*, T^{m^*}\xi^*, t_1) < 1, \quad M(\xi^*, T^{m^*}\xi^*, \psi(t_2)) = 1.$$

Conversely, it can be inferred from

$$\begin{aligned} M(\xi^*, T^{m^*}\xi, \phi(t_2)) &\geqslant \gamma(M(\xi^*, \xi_i, \psi(t_2) - t_0), M(\xi_i, T^{m^*}\xi_i, t_0) \\ &\geqslant \gamma(M(\xi^*, \xi_i, \psi(t_2) - t_0), G_{\xi_0}(\psi^i(t_0))) \\ &\to 1 \quad (i \to \infty) \end{aligned}$$

that

$$\begin{aligned} M(T^{m^*}\xi_i, T^{m^*}\xi^*, t_2) &\geqslant \gamma\{M(\xi^*, \xi_i, \psi(t_2)), M(\xi^*, T^{m^*}\xi_i, \psi(t_2))\} \\ &\to 1 \quad (i \to \infty). \end{aligned}$$

Since

$$\begin{aligned} M(\xi^*, T^{m^*}\xi_i, t_1 - t_2) &\geqslant \gamma\left(M\left(\xi^*, \xi_i, \frac{t_1 - t_2}{2}\right), G_{\xi_0}\left(\psi^i\left(\frac{t_1 - t_2}{2}\right)\right)\right) \\ &\to 1 \quad (i \to \infty), \end{aligned}$$

we have

$$\begin{aligned} M(\xi^*, T^{m^*}\xi^*, t_1) &\geqslant \gamma\left(M(\xi^*, T^{m^*}\xi_i, t_1 - t_2), M(T^{m^*}\xi_i, T^{m^*}\xi^*, t_2)\right) \\ &\to 1 \quad (i \to \infty), \end{aligned}$$

This contradicts the condition $M(\xi' T^{m^*}\xi' t_1) < 1$. Thus, we can conclude that $t_0 = 0$, which implies $M(\xi' T^{m^*}\xi^*, t) = 1$ for all $t > 0$, or in other words, $T^{m^*}\xi^* = \xi^*$.

Next, we aim to prove that $\xi^*$ is the unique fixed point of $T^{m^*}$. Suppose, for the sake of argument, that $\eta^*$ is also a fixed point of $T^{m^*}$. Then, for any $t > 0$, we have:

$$\begin{aligned} M(\xi^*, \eta^*, t) &= M(T^{m^*}\xi^*, T^{m^*}\eta^*, t) \\ &\geqslant \gamma\{M(\xi^*, \eta^*, \psi(t)), M(\xi^*, \xi^*, \psi(t)), M(\xi^*, \eta^*, \psi(t))\} \\ &= M(\xi^*, \eta^*, \psi(t)) \\ &\geqslant \cdots \\ &\geqslant M(\xi^*, \eta^*, \psi^i(t)) \to 1 \quad (i \to \infty), \end{aligned}$$

i.e., $\xi^* = \eta^*$. Thus, the fixed point of $T^{m^*}$ is unique.
Finally, we prove that $\xi^*$ is also the unique fixed point of $T$ and $T^n\xi_0 \to \xi^*$. In fact, since $T^{m^*}\xi^* = \xi^*$, $T^{m^*}T\xi^* = T\xi^*$. Noting that $\xi^*$ is the unique fixed point of $T^{m^*}$, thus we have $\xi^* = T\xi^*$. The uniqueness of the fixed point $\xi^*$ is obvious.
For any $n \in \mathbb{N}$, $n > m^*$, we may write it by $n = km^* + s$, $0 \leqslant s < m^*$. For any $t > 0$, by (2), we have, for all $t > 0$,

$$\begin{aligned} M(\xi^*, T^n\xi_0, t) &= M(T^{m^*}\xi^*, T^{km^*+s}\xi_0, t) \\ &\geqslant \gamma\left\{M(\xi^*, T^{(k-1)m^*+s}\xi^*, \psi(t)), M(\xi^*, T^n\xi_0, \psi(t))\right\} \\ &\geqslant \cdots \\ &\geqslant \gamma\left\{M(\xi^*, T^{(k-1)m^*+s}\xi_0, \psi(t)), M(\xi^*, T^n\xi_0, \psi^i(t))\right\} \\ &\to 1 \quad (k \to \infty). \end{aligned}$$

Letting $i \to \infty$, we obtain, for all $t > 0$,

$$\begin{aligned} M(\xi^*, T^n\xi_0, t) &\geqslant M(\xi^*, T^{(k-1)m^*+s}\xi_0, \psi(t)) \\ &\geqslant \cdots \\ &\geqslant M(\xi^*, T^s\xi_0, \psi^k(t)) \\ &\to 1 \quad (k \to \infty). \end{aligned}$$

This implies that $T^n\xi_0 \to \xi^*$ as $k \to \infty$. This completes the proof. ■

**Theorem 3.5.** *Consider a fuzzy quasi-metric space denoted as $(X, M, \gamma)$, and let the t-norm $\gamma$ satisfy the condition that for any $t_0 \in (0, 1]$, $\gamma(t, t_0)$ is continuous at $t = 1$. Within this framework, we have a mapping $T : X \to X$ that adheres to the following conditions:*

*(i) for each $\xi \in X$, $\sup_{t\in\mathbb{R}} G_\xi(t) = 1$,*
*(ii) For each $\xi \in X$, there exists $m(\xi) \in \mathbb{N}$ such that for all $\eta \in X$ and $t \in \mathbb{R}^+$, the inequality given below holds, where $k \in (0, 1)$ is a constant:*

$$M(T^{m(x)}x, T^{m(x)}y, t) \geqslant \gamma\left\{M\left(x, y, \frac{t}{k}\right), M\left(x, T^{m(x)}y, \frac{t}{k}\right), M\left(x, T^{m(x)}y, \frac{t}{k}\right)\right\},$$

*Under these conditions, it can be concluded that $T$ possesses a unique fixed point $\xi^* \in X$, and for any initial point $\xi_0 \in X$, the iterative sequence $\{T^n\xi_0\}$ converges to the point $\xi_0$.*

*Proof.* Initially, it's important to observe that, based on the assumption regarding $\gamma$, we have $\sup_{t\in(0,1)} \gamma(t, t) = 1$. By considering $\psi(t) = \frac{t}{k}$, we can confirm that it fulfills the conditions outlined in Theorem 3.4.
Subsequently, we aim to establish that, for every $t \in \mathbb{R}^+$,

$$M(\xi^*, T^{m^*}\xi^*, t) = 1.$$

Indeed, for any $t > 0$, we can observe that

$$M(\xi^*, T^{m^*}\xi^*, t) \geqslant \gamma\left(M\left(\xi^*, \xi_i, \frac{t}{2}\right), G_{\xi_0}\left(\frac{t}{2k^i}\right)\right) \to 1 \quad (i \to \infty).$$

Choosing $k_1$ from the interval $(k, 1)$, we obtain $0 < \frac{k}{k_1} < 1$, and

$$\begin{aligned} M\left(T^{m^*}\xi_i, T^{m^*}\xi^*, \frac{k}{k_1}t\right) &\geqslant \gamma\left\{M\left(\xi^*, \xi_i, \frac{t}{k_1}\right)\right. \\ &\quad \left., M\left(\xi^*, T^{m^*}\xi^*, \frac{t}{k_1}\right), M\left(\xi^*, T^{m^*}\xi_i, \frac{t}{k_1}\right)\right\}, \end{aligned}$$

for all $t > 0$. As $M\left(\xi^*, \xi_i, \frac{t}{k_1}\right) \to 1$ and $M\left(\xi^*, T^{m^*}\xi_i, \frac{t}{k_1}\right) \to 1$ as $i \to \infty$, we can deduce that there exists a natural number $n$ such that for $i > N$, the following holds:

$$M\left(T^{m^*}\xi_i, T^{m^*}\xi^*, \frac{k}{k_1}t\right) \geqslant M\left(\xi^*, T^{m^*}\xi^*, \frac{t}{k_1}\right), \quad t > 0.$$

Conversely, we can observe that

$$\begin{aligned} M\left(\xi^*, T^{m^*}\xi^*, t\right) &\geqslant \gamma\left(M\left(\xi^*, T^{m^*}\xi_i, \left(1 - \frac{k}{k_1}\right)t\right), M\left(T^{m^*}\xi_i, T^{m^*}\xi_0, \frac{k}{k_1}t\right)\right) \\ &\geqslant \gamma\left(M\left(\xi^*, T^{m^*}\xi_i, \left(1 - \frac{k}{k_1}\right)t\right), M\left(\xi_i, T^{m^*}\xi^*, \frac{k}{k_1}t\right)\right). \end{aligned}$$

Taking the limit as $i \to \infty$ in the preceding expression, we obtain, for all $t > 0$,

$$\begin{aligned} M\left(\xi^*, T^{m^*}\xi^*, t\right) &\geqslant M\left(\xi^*, T^{m^*}\xi^*, \frac{t}{k_1}\right) \\ &\geqslant \cdots \\ &\geqslant M\left(\xi^*, T^{m^*}\xi^*, \frac{t}{k_1^j}\right) \\ &\to 1 \quad (j \to \infty). \end{aligned}$$

As a result, we find that $M\left(\xi^*, T^{m^*}\xi^*, t\right) = 1$ for all $t > 0$. Referring to Theorem 3.4, we can now draw the desired conclusion. This concludes the proof. ■

**Example 3.6.** *Let $X = \left\{2^{2^n} : n \in \mathbb{N}\right\} \cup \{2\}$ and let $\gamma$ be the usual product. Consider a fuzzy set $M : X^2 \times (0, \infty)$ given by the formula $M(x, y, t) = \frac{t}{t + d(x,y)}$ for each $t > 0$, where*

$$d(x, y) = \begin{cases} \frac{x}{y}, & \text{if } x \leqslant y; \\ \frac{y}{x}, & \text{if } y \leqslant x. \end{cases}$$

*For every positive value of $t$, the triple $(X, M, \gamma)$ forms a complete fuzzy quasi-metric space. Now, let's examine a mapping $\psi$ defined as $\psi(t) = \frac{t}{4}$ for $t$ in the set of positive real numbers, and introduce a function $T : X \to X$ as follows:*

$$T(2^{2^n}) = 2^{2^{n-1}} \quad \text{for } n \in \mathbb{N}, \quad T(2) = 2.$$

*Straightforward computations demonstrate that the conditions stated in Theorem 3.5 are satisfied. Consequently, it can be concluded that $T$ is a fuzzy contractive mapping with the contraction constant $k = 1/2$, and it is evident that the number 2 serves as the exclusive fixed point of $T$.*

## 4. Fixed point Theorems for mappings in fuzzy quasi-metric space and applications

This section aims to introduce the concept of a fuzzy quasi-metric space and investigate whether fixed points exist for mappings in such spaces. We give a number of fixed-point theorems that are applicable to mappings in fuzzy quasi-metric spaces as a concrete example. The results presented in this section supplement and expand on certain recent discoveries found in the literature.

Now let's apply a partial order $''\leqslant''$ on the set $\mathcal{D}^+$ in the way shown below: $\mathcal{D}^+ = \{\psi : \psi \in \mathcal{D}, \psi(t) = 0,, \text{for all } t \leqslant 0\}$, where $\mathcal{D}$ is the set containing all distribution functions.

For any two elements $M_1$ and $M_2$ in $\mathcal{D}^+$, we define $M_1 \leqslant M_2$ if and only if $M_1(t) \geqslant M_2(t)$ for all $t \in \mathbb{R}$. Henceforth, we denote this ordered set as $W = (\mathcal{D}^+, \leqslant)$. It is evident that $W$ satisfies the following conditions:

$(W_1)$ there exists a minimal element $\theta \overset{\text{def}}{=} H \in W$, i.e., $\theta = H \leqslant w$ for all $w \in W$,

$(W_2)$ for any $M_1, M_2 \in W$, the supremum

$$\sup\{M_1, M_2\} \overset{\text{def}}{=} \inf\{M_1(t), M_2(t)\},$$

$(W_3)$ If we define an additive operation on $W'' + '' : W^3 \times W$, i.e., for any $M_1, M_2, M_3, M_4 \in W$, define

$$(M_1 + M_2 + M_3)(t) \overset{\text{def}}{=} \sup_{r+s+l=t} \gamma(M_1(r), M_2(s), M_3(l)),$$

where $\Delta$ is a continuous $t$-norm, it easy to show that $M_1 + M_2 + M_3 \in W$ and

(a) $M_1 + M_2 + M_3 = M_1 + M_3 + M_2 = M_3 + M_1 + M_2, F + \theta = \theta + F = F$,

(b) if $M_1 \leqslant M_2$, then $M_1 + M_3 + M_4 \leqslant M_2 + M_3 + M_4$,

$(W_4)$ if $\{M_n\}$ is a sequence in $W$ and $M_{n+1} \leqslant M_n$, $n \in \mathbb{N}$, then we define the limit operation (denote by $M_n \to M$ or $\lim M_n = M$) having the following properties:

(i) if $M_n = M$, $n \in \mathbb{N}$, then $M_n \to M$,

(ii) if $M_n \to M$, $M'_n \to M'$ and $M''_n \to M''$, then $M_n + M'_n + M''_n \to M + M' + M''$,

(iii) if $M_n \to F, M_n \to F'$, $M_n \leqslant M'_n$, $n \in \mathbb{N}$, then $M \leqslant M'$.

**Definition 4.1.** *Consider a nonempty set $X$ with a self-mapping $T$. We define a point $x \in X$ as a periodic point of $T$ if there exists a positive integer $k$ such that $T^k x = x$. The smallest positive integer that fulfills this condition is referred to as the periodic index of $x$.*

We give some fixed point theorem in a fuzzy quasi-metric spaces.

**Theorem 4.2.** *Let $(X, M)$ be a fuzzy quasi-metric space, and consider a self-mapping $T$ of $X$. If, for any $x \in X$ and for any positive integer $n \geqslant 2$ that satisfies the condition:*

$$T^i x \neq T^j x, \quad 0 \leqslant i < j \leqslant n-1 \tag{5}$$

*we also have:*

$$M(T^n x, T^i x, t) > \min_{1 \leqslant j \leqslant n} M(T^j x, x, t), \quad t > 0, \quad i = 1, 2, \cdots, n-1. \tag{6}$$

*Then, $T$ possesses a fixed point within $X$ if and only if there exist integers $m$ and $n$ with $m > n \geqslant 0$, along with a point $x \in X$ satisfying:*

$$T^m x = T^n x. \tag{7}$$

*If this condition holds, then $T^n x$ serves as a fixed point of $T$ within the space $X$.*

*Proof.* Let $x^* \in X$ be a fixed point of $T$, denoted as $Tx^* = x^*$. In this case, (7) holds true with $m = 1$ and $n = 0$.
Conversely, assuming the existence of a point $x \in X$ and two integers $m$ and $n$, where $m > n \geqslant 0$, satisfying (7), we can establish the following without loss of generality: Let $m$ be the smallest integer for which $T^k x = T^n x$, where $k > n$. We define $y = T^n x$ and $p = m - n$, which leads to $T^p y = y$. Additionally, $p$ is the smallest integer satisfying $T^k y = y$, where $k \geqslant 1$. Next, we aim to demonstrate that $y$ is a fixed point of $T$. Suppose the opposite. In that case, $p \geqslant 2$, and we have:

$$T^i y \neq T^j y, \quad 0 \leqslant i < j \leqslant p-1.$$

By referring to (6), we derive the following:

$$\begin{aligned} M(y, T^i y, t) = M(T^p y, T^i y, t) &> \min_{1 \leqslant j \leqslant p} \{M(T^j y, y, t)\} \\ &\geqslant \min_{1 \leqslant j \leqslant p-1} \{M(T^j y, y, t)\}, \quad i = 1, 2, \cdots, p-1 \text{ and } t > 0. \end{aligned}$$

it follows that

$$\min_{1 \leqslant j \leqslant p} \{M(y, T^j y, t)\} > \min_{1 \leqslant j \leqslant p} \{M(T^j y, y, t)\}, \text{ for all } t > 0$$

This conclusion contradicts our initial assumption. Therefore, we can conclude that $y = T^n x$ is indeed a fixed point of $T$, thus completing the proof. ■

**Theorem 4.3.** *Consider a fuzzy quasi-metric space $(X, M)$ with a self-mapping $T$. Assuming that for any distinct points $x$ and $y$ in $X$, the following condition holds for all $t > 0$:*

$$\begin{aligned} M(Tx, Ty, t) > & \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ & M(x, Ty, t), M(y, Tx, t)\}. \end{aligned} \tag{8}$$

*Under these conditions, the mapping $T$ possesses a fixed point in $X$ if and only if there exists a periodic point $x$ for $T$. Furthermore, when this condition is met, the point $x$ is the unique fixed point of $T$ within $X$.*

*Proof.* Suppose that we have a periodic index $k$ for the point $x$, and we define a set $A$ as follows:

$$A = \{x, Tx, \cdots, T^{k-1}x\}.$$

Now, let's assume that $x$ is not a fixed point of $T$. In this case, all the points in set $A$ must be distinct. Consequently, for any pair of integers $i$ and $j$ from the set $0, 1, 2, \cdots, k-1$ where $i \neq j$, we can conclude that:

$$T^{i-1}x \neq T^{j-1}x \quad (x = T^k x, \text{ as } i = 0 \text{ or } j = 0).$$

Now, utilizing the condition stated in (8), we can deduce the following inequality:

$$\begin{aligned} M(T^i x, T^j x, t) &> \min\{M(T^{i-1}x, T^{j-1}x, t), M(T^{i-1}x, T^i x, t), M(T^{j-1}x, T^j x, t), \\ &\quad M(T^{i-1}x, T^j x, t), M(T^{j-1}x, T^i x, t)\} \\ &> \min_{0 \leqslant i,j \leqslant k-1} \min\{M(T^i x, T^j x, t)\} \text{ for all } t > 0. \end{aligned}$$

Applying property (W4), we obtain:

$$\min_{0 \leqslant i,j \leqslant k-1} \min\{M(T^i x, T^j x, t)\} > \min_{0 \leqslant i,j \leqslant k-1} \min\{M(T^i x, T^j x, t)\},$$

which is a contradiction. Therefore, we conclude that $x = Tx$. It is evident that $x$ is the unique fixed point of $T$ within $X$. This concludes the proof of the sufficiency condition. The necessity condition is self-evident. Thus, the proof is complete. ■

The subsequent theorem can be readily derived from Theorem 4.3.

**Theorem 4.4.** *Consider a fuzzy quasi-metric space $(X, M)$ with a self-mapping $T$. Assuming the existence of a positive integer $p$ such that, for any distinct points $x$ and $y$ in $X$ and for all $t > 0$, the following condition holds:*

$$\begin{aligned} M(T^p x, T^p y, t) \quad > \quad & \min\{M(x, y, t), M(x, T^p x, t), M(y, T^p y, t), \\ & M(x, T^p y, t), M(y, T^p x, t)\}. \end{aligned} \tag{9}$$

*Under these conditions, the mapping $T$ has a fixed point in $X$ if and only if there exists a periodic point $x$ for $T$. Moreover, when this condition is met, the point $x$ is the unique fixed point of $T$ within $X$.*

**Theorem 4.5.** *Consider a fuzzy quasi-metric space $(X, M)$ with a self-mapping $T$. Assuming the existence of positive integers $p$ and $q$, for any distinct points $x$ and $y$ in $X$ and for all $t > 0$, the following inequality holds:*

$$\begin{aligned} M(T^p x, T^q y, t) \quad > \quad & \min\{M(x, y, t), M(x, T^p x, t), M(y, T^q y, t), \\ & M(x, T^q y, t), M(y, T^p x, t)\}. \end{aligned} \tag{10}$$

*Under these conditions, the mapping $T$ has a fixed point in $X$ if and only if there exists a periodic point $x$ of $T$ with a periodic index $k$ that satisfies the following condition:*

$$k \neq 2|p_2 - q_2|, \tag{11}$$

*Here, $p = p_1 k + p_2$ and $q = q_1 k + q_2$, where $0 \leqslant p_2, q_2 < k$, and $p_1, q_1$ are non-negative integers. If this condition is met, then this point $x$ serves as the unique fixed point of $T$ within $X$.*

*Proof.* The necessity condition is self-evident.

Now, for the sufficiency part: Let's assume that $k$ is the periodic index of the point $x$, and define the set $A$ as follows:

$$A = \{x, Tx, \cdots, T^{k-1}x\}.$$

Suppose that $x$ is not a fixed point of $T$. In this case, all the points in set $A$ must be distinct. For any integers $i$ and $j$, where $0 \leqslant i < j < k$, due to the properties $T^{p_2}(A) = A$ and $T^{q_2}(A) = A$, we can find $T^{n_1}x$, $T^{n_2}x$, $T^{m_1}x$, and $T^{m_2}x$ within $A$ such that the following conditions are satisfied:

$$T^{p_2}(T^{n_1}x) = T^i x, \quad T^{q_2}(T^{n_2}x) = T^j x, \tag{12}$$

$$T^{q_2}(T^{m_1}x) = T^i x, \quad T^{p_2}(T^{m_2}x) = T^j x \tag{13}$$

Now, let's prove that at least one of the statements $T^{n_1}x \neq T^{n_2}x$ or $T^{m_1}x \neq T^{m_2}x$ is true. Suppose for the sake of contradiction that both of these statements are false. This would mean that $n_1 = n_2$ and $m_1 = m_2$. By using (12) and (13), we can derive the following equations:

$$p_2 + n_1 = a_1 k + i, \quad q_2 + n_1 = a_2 k + j, \tag{14}$$

$$q_2 + m_1 = b_1 k + i, \quad p_2 + m_1 = b_2 k + j, \tag{15}$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are in the range $(0, 1)$. Without loss of generality, let's assume that $p_2 > q_2$. From (14) and (15), and considering that $i < j$, it follows that $a_1 = 1$, $a_2 = 0$, and:

$$(1 + b_1 - b_2)k = 2(j - i). \tag{16}$$

Since $0 \leqslant i < j < k$, we must have $b_1 = b_2$. From (14)-(16), it becomes evident that $k = 2(p_2 - q_2)$, which contradicts (11). Without loss of generality, we can assume that $T^{n_1}x \neq T^{n_2}x$. Since $T^i x = T^{p_2}(T^{n_1}x) = T^p(T^{n_1}x)$ and $T^j x = T^{q_2}(T^{n_2}x) = T^q(T^{n_2}x)$, it follows from (10) that:

$$\begin{aligned} M(T^i x, T^j x, t) \quad > \quad & \min\{M(T^{n_1}x, T^{n_2}x, t), M(T^{n_1}x, T^i x, t), M(T^{n_2}x, T^j x, t) \\ & , M(T^{n_1}x, T^j x, t), M(T^{n_2}x, T^i x, t)\} \\ > \quad & \min_{0 \leqslant i, j \leqslant k-1} \{M(T^i x, T^j x, t)\} \end{aligned}$$

for all $t > 0$. Using property (W4), we then arrive at the contradiction:

$$\min_{0\leqslant i,j\leqslant k-1}\{M(T^ix,T^jx,t)\} > \min_{0\leqslant i,j\leqslant k-1}\{M(T^ix,T^jx,t)\}$$

This contradiction leads us to conclude that $x = Tx$. Furthermore, it is evident that $x$ is the unique fixed point of $T$ in $X$. This concludes the proof. ∎

The proofs of the following results share a similar approach, so we will skip the detailed proof.

**Theorem 4.6.** *Let $(X, M)$ be a fuzzy quasi-metric space, and let $T$ be a self-mapping of $X$. Assume that for every $x \in X$, there exists a positive integer $p(x)$ such that for all $y \in X$, where $x \neq y$, and for every $t > 0$, the following inequality holds:*

$$\begin{aligned} M(T^{p(x)}x,T^{p(x)}y,t) \quad > \quad & \min\{M(x,y,t),M(x,T^{p(x)}x,t),M(y,T^{p(x)}y,t),\\ & M(x,T^{p(x)}y,t),M(y,T^{p(x)}x,t)\}. \end{aligned}$$

*Under these conditions, the mapping $T$ has a fixed point in $X$ if and only if there exists a periodic point $x$ of $T$ with a periodic index $k$. For this point $x$, which belongs to the set $A = \{x, Tx, \dots, T^{k-1}x\}$, the following conditions must be satisfied: for any pair of distinct elements $u$ and $v$ in $A$, there exist $x'$ and $y'$ in $A$, where $x'$ is not equal to $y'$, and they satisfy the following equations:*

$$T^{p(x')}x' = u, \quad T^{p(x')}y' = v.$$

*If these conditions are met, then point $x$ is the unique fixed point of $T$ within $X$.*

**Theorem 4.7.** *Consider a fuzzy quasi-metric space $(X, M)$ with a self-mapping $T$. Suppose that for any given $x \in X$ where $x \neq y$, there exists a positive integer $p(x, y)$ such that for all $t > 0$, the following inequality holds:*

$$\begin{aligned} M(T^{p(x,y)}x,T^{p(x,y)}y,t) \quad > \quad & \min\{M(x,y,t),M(x,T^{p(x,y)}x,t),M(y,T^{p(x,y)}y,t),\\ & M(x,T^{p(x,y)}y,t),M(y,T^{p(x,y)}x,t)\}. \end{aligned}$$

*Under these conditions, the mapping $T$ has a fixed point in $X$ if and only if there exists a periodic point $x$ of $T$ with a periodic index $k$. For this point $x$, which belongs to the set $A = x, Tx, \dots, T^{k-1}x$, the following conditions must be satisfied: for any pair of distinct elements $u$ and $v$ in $A$, there exist $x'$ and $y'$ in $A$, where $x'$ is not equal to $y'$, and they satisfy the following equations:*

$$T^{p(x',y')}x' = u, \quad T^{p(x',y')}y' = v.$$

*If these conditions are met, then point $x$ is the unique fixed point of $T$ within $X$.*

## 5. Fixed point theorems of Kransnoselski's type in fuzzy quasi-metric spaces

In functional and nonlinear analysis, Krasnoselskii's fixed-point theorems are essential because they specify the requirements for a mapping to have a fixed point in a space. These theorems are modified to include parameters in parameterized metric spaces (PM-spaces), and the continuity of fixed points with respect to these parameters is important in mathematical modeling, control theory, and optimization. In physics, it helps to understand phase transitions; in economics, it helps to understand equilibria when economic factors change; and in control systems, it helps to evaluate stability and performance. The significance of fixed-point theorems in a variety of applications is examined in this section as it relates to fuzzy quasi-metric spaces.

Several fixed point theorems of Kransnoselski's nature in the framework of FM-space will be presented in this section. We will also discuss how fixed points show continuity with respect to changing factors.

To begin, let's demonstrate the following definition.

**Definition 5.1.** *Let $(X, N, \gamma)$ be a fuzzy quasi-normed space equipped with a continuous t-norm $\gamma$, and let $A$ be a nonempty subset of $X$. A mapping $T : X \to X$ is said to be compact if $\overline{T(A)}$ is a compact subset of $X$.*

**Lemma 5.2.** *Consider a complete fuzzy quasi-metric space denoted as* $(X, M, \gamma)$*, with the added condition that the family* $\{\Psi_n(\gamma, u)\}_{n\in\mathbb{N}}$ *exhibits equicontinuity at the point* $u = 1$*. Now, suppose we have a mapping* $\varphi : X \to X$*, and for each element* $x \in X$*, there exists a natural number* $n(x) \in \mathbb{N}$ *such that, for all* $y \in X$ *and* $u > 0$*, the following inequality holds:*

$$M(\varphi^{n(x)}(x), \varphi^{n(x)}(y), ku) \geqslant M(x, y, u),$$

*where* $k \in (0, 1)$*. Additionally, define* $\Psi_1(\gamma, u)$ *as* $\gamma(u, u)$ *and* $\Psi_n(\gamma, u)$ *as:*

$$\Psi_n(\gamma, u) = \underbrace{\gamma(\gamma(\cdots\gamma}_{n-times}(\gamma(u, u), u,), \cdots, u)),\ n = 2, 3, \cdots,,, u \in [0, 1].$$

*Under these conditions, it can be concluded that* $\varphi$ *possesses a unique fixed point denoted as* $x^*$*, and for every* $x \in X$*,* $\lim_{n\to\infty} \varphi^n(x) = x^*$*.*

*Proof.* By Theorem 3.5, it suffices to prove that for every $x_0 \in X$,

$$\sup_u G_{x_0}(u) = 1,$$

where $G_{x_0}(u) = \inf_{n\in\mathbb{N}} M(x_0, \varphi^n(x_0), u)$.
Let $m \in \mathbb{N}$ and $sn(x_0) < m \leqslant (s + 1)n(x_0)$. Then for every $u > 0$,

$$\begin{aligned}
&M(\varphi^n(x_0), x_0, u) \geqslant \gamma(M(\varphi^n(x_0), \varphi^{n(x_0)}(x_0), ku), M(\varphi^{n(x_0)}(x_0), x_0, u - ku))\\
&\geqslant \gamma(M(\varphi^{m-n(x_0)}(x_0), x_0, u), M(\varphi^{n(x_0)}(x_0), x_0, u - ku))\\
&\geqslant \underbrace{\gamma(\gamma(\cdots\gamma}_{n-\text{times}}(M(\varphi^{m-n(x_0)}(x_0), x_0, u), M(\varphi^{n(x_0)}(x_0), x_0, u - ku)),\\
&\cdots, M(\varphi^{n(x_0)}(x_0), x_0, u - ku))).
\end{aligned}$$

If $g(u) = \min_{r=1,2,\cdots,n(x_0)}\{M(\varphi^r(x_0), x_0, u - ku)\}$, then we have

$$M(\varphi^m(x_0), x_0, u) \geqslant \Psi_s(t, g(u)),\ m \in \mathbb{N}.$$

Since $\lim_{u\to\infty} g(u) = 1$ and the family $\{\Psi_s(t, u)\}_{n\in\mathbb{N}}$ is equicontinuous,

$$M(\varphi^m(x_0), x_0, u) > 1 - \lambda$$

for every $m \geqslant n(u, \lambda)$ and $\lambda \in (0, 1)$ and so $\sup_u G_{x_0}(u) = 1$. This complete the proof. ∎

Subsequently, we provide an illustration of a $t$-norm $\gamma$ where the family $\{\psi_s(t, u)\}_{s\in\mathbb{N}}$ demonstrates equicontinuity specifically at the point $u = 1$.

**Example 5.3.** *Consider a continuous t-norm denoted as* $\widetilde{\gamma}$ *and a set of intervals* $I_m = [1 - 2^m, 1 - 2^{-m-1}]$ *for* $m = 0, 1, 2, \ldots$*. If we define the mapping* $\gamma : [0, 1] \times [0, 1] \to [0, 1]$ *as follows:*

$$\begin{cases} 1 - 2^{-m} + 2^{-m-1}\widetilde{\gamma}(2^{m+1}(x - 1 + 2^{-m}), & \\ 2^{m+1}(y - 1 + 2^{-m})), & (x, y) \in \bigcup_{m\in\mathbb{N}\cup\{0\}} I_m^2, \\ \min\{x, y\}, & (x, y) \notin \bigcup_{m\in\mathbb{N}\cup\{0\}} I_m^2, \end{cases}$$

*then the family* $\{\psi_s(t, u)\}_{s\in\mathbb{N}}$ *demonstrates equicontinuity specifically at the point* $u = 1$*.*

The subsequent lemma holds significant utility in the upcoming sections, and its details can be referenced in [10].

**Lemma 5.4.** *Let $X$ represent a Hausdorff topological vector space, $K$ be a nonempty, closed, and convex subset of $X$, $\varphi : X \to X$ be an affine and continuous mapping, and $\psi : K \to X$ be a continuous mapping. Additionally, assume that $\overline{\psi(K)}$ is compact, and the following conditions hold:*

*(a) For any $y$ within the closure of the convex hull of $\psi(K)$ (denoted as $\overline{co}\psi(K)$), there is precisely one solution, denoted as $x(y)$, in $K$ for the equation $z = \varphi z + y$. Furthermore, the set $\{x(y)\}_{y \in \overline{\psi(K)}}$ is compact.*

*(b) For every $V$ belonging to the collection $\mathcal{U}$ of neighborhoods of zero in $X$ and every $x$ within the closure of $\psi(K)$, there exists a $U$ in $\mathcal{U}$ such that $co(x+U) \cap \overline{\psi(K)} \subseteq x + V$.*

*Under these circumstances, there exists a point $x$ within the set $K$ such that $x = \varphi x + \psi x$.*

Utilizing Lemma 5.4, we aim to establish a fixed point theorem in the style of Kransnoselski for fuzzy quasi-normed spaces.

**Theorem 5.5.** *Consider a complete fuzzy quasi-normed space $(X, N, \gamma)$ equipped with a continuous t-norm $\gamma$. Let $K$ be a closed, convex, and fuzzy bounded subset of $X$, and let $\varphi : X \to X$ be a linear continuous mapping. Additionally, consider a continuous mapping $\vartheta : K \to X$ such that $\overline{\vartheta(K)}$ is compact, and ensure the following conditions are met:*

*(a) There exists $n \in \mathbb{N}$ and $k \in (0,1)$ such that for any $\varepsilon > 0$ and all $x, y \in K$, the inequality holds:*

$$N(\varphi^n(x) - \varphi^n(y), k\varepsilon) \geqslant N(x - y, \varepsilon).$$

*(b) The set inclusion $\varphi(K) + \overline{co}(K) \subseteq K$ is satisfied.*

*(c) For any $\varepsilon > 0$, $\delta \in (0,1)$, and $x \in \vartheta(K)$, there exist $\varepsilon' > 0$ and $\delta' \in (0,1)$ such that:*

$$co(x + U(\varepsilon', \delta')) \cup \vartheta(K) \subseteq x + U(\varepsilon, \delta).$$

*Under these conditions, there exists a point $x \in K$ such that $x = \varphi x + \vartheta x$.*

*Proof.* From Theorem 6 in reference [14], we can deduce that for every $y \in \overline{co}(K)$, there exists an element $\eta_y \in K$ satisfying the equation $\eta_y = \varphi(\eta_y) + y$. This result implies the existence of a mapping $\Gamma_y : K \to K$ defined as $\Gamma_y(x) = \varphi x + y$ for $y \in \overline{co}(K)$ and $x \in K$. Notably, this mapping $\Gamma_y$ satisfies the condition $\Gamma_y^n(x) - \Gamma_y^n(z) = \varphi^n(x) - \varphi^n(z)$ for $x, z \in K$, given the linearity of $\varphi$ and the fact that $K$ is a fuzzy bounded subset of $X$, which implies that $\sup_x D_K(x) = 1$, where $D_K(x) = \sup_{t<x} \inf_{p,q \in K} N(p - q, t)$ for $x \in \mathbb{R}$.

Hence, there exists a unique element $\eta_y$ such that $\Gamma_y(\eta_y) = \eta_y$ for $y \in \overline{co}(K)$. Now, we aim to demonstrate that the mapping $\eta : \overline{\vartheta(K)} \to K$ is continuous. Let us denote the sets of all continuous mappings from $X$ and $K$ as $C(\overline{\vartheta(K)}, X)$ and $C(\overline{\vartheta(K)}, K)$, respectively. Then, the triplet $(C(\overline{\vartheta(K)}, X), \widetilde{N}, \gamma)$ forms a complete fuzzy quasi-normed space, where the mapping $\widetilde{N} : C(\overline{\vartheta(K)}, X) \to \mathcal{D}^+$ is defined as follows:

$$\widetilde{N}(\tilde{x}, t) = \sup_{\delta < t} \inf_{y \in \overline{\vartheta(K)}} N(\tilde{x}(y), \delta)$$

for every $\tilde{x} \in C(\overline{\vartheta(K)}, X)$. If we define the mapping

$$\widehat{\Gamma} : C(\overline{\vartheta(K)}, K) \to C(\overline{\vartheta(K)}, X)$$

as $(\widehat{\Gamma}\tilde{x})(y) = \varphi(\tilde{x}(y)) + y$ for every $y \in \overline{\vartheta(K)}$ and $\tilde{x} \in C(\overline{\vartheta(K)}, K)$, then we can express

$$(\widehat{\Gamma}^n \tilde{x})(y) = \varphi^n(\tilde{x}(y)) + \sum_{j=0}^{n-1} \varphi^j y,$$

and it's straightforward to observe that

$$\widetilde{N}(\widehat{\Gamma}^n \tilde{x}_1 - \widehat{\Gamma}^n \tilde{x}_2, ku) \geqslant \widetilde{N}(\tilde{x}_1 - \tilde{x}_2, u), , , u > 0$$

for every $\tilde{x_1}, \tilde{x_2} \in C(\overline{\vartheta(K)}, X)$. This is due to the fact that

$$\sup_{\varepsilon>0} \inf_{\tilde{x_1},\tilde{x_2}\in C(\overline{\vartheta(K)},K)} \tilde{N}(\tilde{x_1} - \tilde{x_2}, \varepsilon) = 1,$$

as established in [14, Theorem 6]. Consequently, it follows that there exists a unique element $\tilde{x}^* \in C(\overline{\vartheta(K)}, K)$ such that $\hat{\Gamma}\tilde{x}^* = \tilde{x}^*$. Subsequently, $\tilde{x}^*(y) = \eta_y$ for every $y \in \overline{\vartheta(K)}$, and this leads to the conclusion that the mapping $\eta$ is continuous.

Moreover, since the set $\overline{\vartheta(K)}$ is compact, condition (a) from Lemma 5.4 is satisfied. This completes the proof. ■

**Corollary 5.6.** *Let's consider a complete fuzzy quasi-normed space $(X, N, \gamma)$ equipped with a continuous t-norm $\gamma$. Additionally, assume that the family $\{\Psi_n(\gamma, u)\}_{n\in\mathbb{N}}$ exhibits equicontinuity at the point $u = 1$. We also have a linear continuous mapping $\varphi : X \to X$ and a compact mapping $\vartheta : K \to X$ that satisfy conditions (a) and (b) in Theorem 5.5. Under these conditions, there exists a point $x \in K$ such that $x = \varphi x + \vartheta x$.*

*Proof.* We will now establish that condition (c) in Theorem 5.5 is indeed satisfied. Let's assume that $\varepsilon > 0$ and $\delta \in (0, 1)$. Our objective is to demonstrate the validity of the following inclusion:

$$\mathrm{co}U(\varepsilon, \delta') \subseteq U(\varepsilon, \delta), \tag{17}$$

where $\delta' \in (0, 1)$ is chosen to ensure the following implication:

$$u \geqslant 1 - \delta' \Rightarrow \Psi_n(\gamma, u) > 1 - \delta, \;\; n \in \mathbb{N}.$$

This implication holds due to the equicontinuity of the family $\Psi_n(\gamma, u)_{n\in\mathbb{N}}$ at the point $u = 1$ and the existence of such a $\delta'$. Now, let's take an arbitrary point $x \in \mathrm{co}\, U(\varepsilon, \delta')$. This means that there exist $\lambda_i \in [0, 1]$, $x_i \in U(\varepsilon, \delta')$ for $i = 1, 2, \cdots, n$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $x = \sum_{i=1}^{n} \lambda_i x_i$. We can proceed with the following calculations:

$$\begin{aligned}
N(x, \varepsilon) &\geqslant N\left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i \varepsilon\right) \\
&\geqslant \gamma\left(N\left(\sum_{i=1}^{n-1} \lambda_i x_i, \sum_{i=1}^{n-1} \lambda_i \varepsilon\right), N(\lambda_n x_n, \lambda_n \varepsilon)\right) \\
&= \gamma\left(N\left(\sum_{i=1}^{n-1} \lambda_i x_i, \sum_{i=1}^{n-1} \lambda_i \varepsilon\right), N(x_n, \varepsilon)\right) \\
&\geqslant \gamma\left(\gamma\left(N\left(\sum_{i=1}^{n-2} \lambda_i x_i, \sum_{i=1}^{n-2} \lambda_i \varepsilon\right), N(x_{n-1}, \varepsilon)\right), N(x_n, \varepsilon)\right) \\
&\geqslant \underbrace{\gamma(\gamma(\cdots\gamma(\gamma}_{n-\text{times}}(N(x_1, \varepsilon), N(x_2, \varepsilon), N(x_3, \varepsilon), \cdots, N(x_n, \varepsilon)))) \\
&\geqslant \Psi_n(\gamma, 1 - \delta') > 1 - \delta,
\end{aligned}$$

which implies that $x \in U(\varepsilon, \delta)$. Thus, we have successfully established the inclusion in equation (17). This concludes the proof. ■

The consistency of stationary points with respect to different parameters will then be examined. Suppose we have a metrizable topological space $C$, a subset $\Delta \subset C$, a mapping $T : E \times \Delta \to X$, where $E$ is a subset of $X$, and a full fuzzy quasi-metric space $X$. Moreover, suppose that the mapping $T_\lambda x \to T(x, \lambda)$ has a fixed point $x(\lambda) \in E$ for all $\lambda \in \Delta$. Next, we want to prove a theorem showing that the fixed points of the mappings $T_\lambda$ depend continuously on the parameter $\lambda \in \Delta$.

Let $A$ be a fuzzy bounded subset of the fuzzy quasi-metric space $(X, M, \gamma)$, and let $\gamma$ be the continuous $t$-norm. The definition of the function $\alpha_A$ is as follows:

$$\begin{aligned} \alpha_A(x) &= \sup\{\varepsilon : \varepsilon > 0, \quad \text{there exists a covering } \{A_j\}_{j\in J} \\ &= J \text{ is finite, } A \subset \bigcup_{j\in J} A_j, D_{A_j}(x) \geqslant \varepsilon\}, \; x \in \mathbb{R} \end{aligned}$$

This function is commonly referred to as Kuratowski's fuzzy number or Kuratowski's function (see [18, Chapter X, page 242]). Kuratowski's function possesses the following properties:

(i) If $\varnothing \neq A \subseteq B \subseteq X$, then $\alpha_A(x) \geqslant \alpha_B(x)$ for all $x \in \mathbb{R}$.
(ii) $\alpha_A(x) = \alpha_{\bar{A}}(x)$ for all $x \in \mathbb{R}$.
(iii) A subset $A$ in $X$ is compact if and only if $\alpha_A(x) = H(x)$ for all $x \in \mathbb{R}$, where $H(x)$ is another function defined by

$$H(x) = \begin{cases} 1, & \text{if } t > 0; \\ 0, & \text{if } t \leqslant 0. \end{cases}$$

**Definition 5.7.** *Consider a topological space denoted as $Y$, a subset $\Delta$ of another topological space $S$, and a mapping $\Phi : \Delta \to 2^Y$. We designate the multifunction $\Phi$ as "upper semicontinuous" at a specific point $\lambda_0 \in \Delta$ if and only if the following condition holds: For every open set $G$ in $Y$ that encompasses $\Phi(\lambda_0)$, there exists an open neighborhood $U(\lambda_0)$ of $\lambda_0$ in $S$ such that $\Phi(U(\lambda_0) \cap \Delta) \subseteq G$.*

**Theorem 5.8.** *Assume we have a complete fuzzy quasi-metric space denoted as $(X, M, \gamma)$ with a continuous t-norm $\gamma$. Additionally, consider a complete metrizable topological space $\mathbf{C}$, a subset $\Delta$ within $\mathbf{C}$, and a closed, fuzzy bounded subset $E$ of $X$. We are given a mapping $T : E \times \Delta \to X$ that satisfies the following conditions:*

*(a) The function $T(\cdot, \lambda)$ is continuous for each $\lambda \in \Delta$, and there exists a subset $\Delta_0 \subset \Delta$ such that for every $\lambda_0 \in \Delta_0$, $T(\cdot, \lambda)$ is continuous at $(x, \lambda_0)$ for every $x \in E$.*
*(b) For each $\lambda \in \Delta$, the equation $x = T(x, \lambda)$ has a solution in $E$.*
*(c) For every subset $E' \subset E$ where $\alpha_{E'} < H$, there exists an open neighborhood $B = B(E)$ of $\Delta_0$ such that, for any precompact set $\Delta' \subset \Delta \cup B$, the inequality $\alpha_{T(E',\Delta')} > \alpha_{E'}$ holds.*

*Under these conditions, we can conclude that $\Phi$ is upper semicontinuous at each $\lambda \in \Delta_0$, where $\Phi(\lambda)$ is defined as the set of solutions to the equation*

$$\Phi(\lambda) = \{x : x \in E, x = T(x, \lambda)\}.$$

*Proof.* Assume that $\Phi$ does not exhibit upper semicontinuity at a specific point $\lambda_0 \in \Delta$. In such a case, there exists an open set $\mathcal{O}$ containing $\Phi(\lambda_0)$, and for any given $\varepsilon > 0$ and $\delta$ within the interval $(0, 1)$, we can find a $\lambda_{\varepsilon,\delta} \in \Delta$ such that:

$$M(\lambda_{\varepsilon,\delta}, \lambda_0, \varepsilon) > 1 - \delta, \quad \text{and} \quad \Phi(\lambda_{\varepsilon,\delta}) \not\subset \mathcal{O}.$$

This means that for any small positive $\varepsilon$ and $\delta$ values, we can identify a $\lambda_{\varepsilon,\delta}$ in $\Delta$ such that $\Phi$ exceeds a threshold of $1 - \delta$ at $\varepsilon$ distance from $\lambda_0$, and the set of points within $\Phi(\lambda_{\varepsilon,\delta})$ is not entirely contained within the open set $\mathcal{O}$.

Let's assume we have a sequence of decreasing positive values: $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots$ with $\lim_{n\to\infty} \varepsilon_n = 0$, and another sequence of decreasing values: $\delta_1 > \delta_2 > \cdots > \delta_n > \cdots$ with $\lim_{n\to\infty} \delta_n = 1$. If we define $\lambda'_n = \lambda_{\varepsilon_n,\delta_n}$, then we can conclude that $\lim_{n\to\infty} \lambda'_n = \lambda_0$. Indeed, consider any $\varepsilon > 0$ and $\delta \in (0, 1)$. There exists some positive integer $n_0$ such that $\varepsilon_n < \varepsilon$ and $\delta < \delta_n$ for all $n \geqslant n_0$. Consequently,

$$M(\lambda'_n, \lambda_0, \varepsilon) \geqslant M(\lambda'_n, \lambda_0, \varepsilon_n), \quad n \geqslant n_0.$$

Since $M(\lambda'_n, \lambda_0, \varepsilon_n) > 1 - \delta_n$ for all $n \in \mathbb{N}$ and $\delta < \delta_n$ for all $n \geqslant n_0$, it follows that

$$M(\lambda'_n, \lambda_0, \varepsilon) \geqslant 1 - \delta, \quad n \geqslant n_0.$$

This implies that $\lim_{n\to\infty} \lambda'_n = \lambda_0$. Now, let's consider the fact that $\Phi(\lambda'_n) \not\subset O$ for each $n \in \mathbb{N}$. Therefore, we can find $x_n \in \Phi(\lambda'_n)\backslash O$ for each $n \in \mathbb{N}$. Define a set $E' = \{x_n : n \in \mathbb{N}\}$. To show that the set $E'$ is precompact, let's assume the opposite, that is, $\alpha_{E'} < H$. For this, we choose a positive integer $k$ such that

$$\Delta' = \{\lambda'_n : n \geqslant k\} \subset B(E').$$

Now, $\Delta'$ is precompact. Since $x_n \in \Phi(\lambda'_n)$ for every $n \in \mathbb{N}$, it follows that $x_n = T(x_n, \lambda'_n)$ for all $n \in \mathbb{N}$. This implies that

$$\alpha_{E'} = \alpha_{x_n:n\geqslant k} \geqslant \alpha_{T(E',\Delta')} > \alpha_{E'}.$$

So, we conclude that $\alpha_{E'} = H$, which means that $\bar{E}'$ is a compact set. As we have $\{x_n\} \subset E' \subset \bar{E}'$, it implies the existence of a subsequence, which we'll still denote as $\{x_n\}$, converging to a point $x_0 \in \bar{E}'$. Consequently, $\{(x_n, \lambda_n)\} \to (x_0, \lambda_0)$. Therefore, $\{T(x_n, \lambda_n)\} \to T(x_0, \lambda_0)$. Thus, we can deduce that $x_0 = T(x_0, \lambda_0)$, which means that $x_0$ belongs to the set $\Phi(\lambda_0)$. However, for each $n \in \mathbb{N}$, it is known that $x_n$ is not in $C$. Since $C^c$, the complement of $C$, is a closed set, it follows that $x_0$ is also not in $C$. This, in turn, implies that $x_0$ belongs to $\Phi(\lambda_0)$. Therefore, we have reached a contradiction. This concludes the proof. ∎

To illustrate Theorem 5.8 with an example involving fractional differential equations, let's examine a specific fractional differential equation and show how it aligns with the framework of a fuzzy quasi-metric space.

**Example 5.9.** *(***Fractional Differential Equation***) Consider the following fractional differential equation involving the Caputo fractional derivative of order $\alpha$, where $0 < \alpha < 1$:*

$$\mathcal{D}^\alpha x(t) = -kx(t)$$

*with the initial condition $x(0) = x_0$. Here, $k$ is a positive constant. This type of fractional differential equation can be used to model various physical processes with memory and hereditary properties.*

*(i) Space X and Fuzzy Quasi-Metric M:*
- *Let $X$ be the space of continuous functions $C([0, 1], \mathbb{R})$.*
- *Define the fuzzy quasi-metric $M$ on $X$ as follows:*

$$M(x, y, t) = \frac{t}{t + \|x - y\|_\infty},$$

*where $\|x - y\|_\infty = \sup_{s\in[0,1]} |x(s) - y(s)|$.*
- *the t-norm $\gamma$ is given by $\gamma(a, b) = \min\{a, b\}$.*

*(ii) topological Space $C$ and Subset $\Delta$:*
- *Let $C$ be the space of continuous functions on $[0, 1]$.*
- *Let $\Delta$ be a compact interval of possible values for the parameter $k$, say $\Delta = [k_1, k_2] \subset \mathbb{R}$.*

*(iii) Set E: Let $E \subset X$ be a closed, fuzzy bounded subset of $C([0, 1], \mathbb{R})$. For simplicity, consider $E$ to be the set of functions bounded by a certain norm, say*

$$E = \{x \in C([0, 1], \mathbb{R}) : \|x\|_\infty \leqslant M\}$$

*for some $M > 0$.*

*(iv) Mapping T: Define the mapping $T : E \times \Delta \to X$ by*

$$(T(x, k))(s) = x_0 E_\alpha(-ks^\alpha),$$

*where $E_\alpha$ is the Mittag-Leffler function, which is the solution to the given fractional differential equation.*

*Verifying Conditions of the Theorem:*
*Condition (a):*

- *The function* $T(\cdot, \lambda)$ *is continuous for each* $\lambda \in \Delta$.
- *For a subset* $\Delta_0 \subset \Delta$, $T(\cdot, \lambda)$ *is continuous at* $(x, \lambda_0)$ *for every* $x \in E$.

*Condition (b): For every subset* $E' \subset E$ *where* $\alpha_E < H$, *there exists an open neighborhood* $B = B(E)$ *of* $\Delta_0$ *such that for any precompact set* $\Delta' \subset \Delta \cup B$, *the inequality* $\alpha_{T(E',\Delta')} > \alpha_{E'}$ *holds.*
*Conclusion: Under these conditions, we can conclude that* $\Phi$ *is upper semicontinuous at each* $\lambda \in \Delta_0$, *where* $\Phi(\lambda)$ *is defined as the set of solutions to the equation:*

$$\Phi(\lambda) = \{x : x \in E, x = T(x, \lambda)\}.$$

*Thus, in the context of this fractional differential equation, the theorem guarantees that the set of solutions* $\Phi(\lambda)$ *will be upper semicontinuous at each* $\lambda \in \Delta_0$. *This means that small changes in the parameter* $\lambda$ *will not cause abrupt changes in the solution set, ensuring the stability of solutions under parameter variations.*

Utilizing Theorem 5.8, we will now establish a theorem regarding the existence of a solution for the system given by $x = H(x, y)$ and $y = K(x, y)$.

**Theorem 5.10.** *Assume that* $(X_1, N_1, \gamma_1)$ *and* $(X_2, N_2, \gamma_2)$ *are both complete fuzzy quasi-normed spaces equipped with continuous t-norms* $\gamma_1$ *and* $\gamma_2$. *Additionally, consider that* $U$ *is a closed and fuzzy bounded subset of* $X_1$, *and* $V$ *is a closed and convex subset of* $X_2$. *We have two mappings,* $H : U \times V \to U$ *and* $K : U \times V \to V$, *which are both compact mappings, and they satisfy the following conditions:*

*(a) For every element* $v$ *in the set* $V$, *the equation* $u = H(u, v)$ *possesses a unique solution, denoted as* $u(v)$, *which belongs exclusively to the set* $U$.
*(b) For every* $v \in V$, *the function* $H(\cdot, v)$ *exhibits continuity, and for each* $u \in U$, *the function* $H(u, \cdot)$ *also demonstrates continuity.*
*(c) For any subset* $U' \subset U$ *with* $\alpha_{U'} < H$ *and any precompact set* $V' \subset V$, *we can deduce that:*

$$\alpha_{H(U',V')} > \alpha_{U'}.$$

*(d) For any given* $\varepsilon > 0$, $\delta \in (0, 1)$, *and any element* $x$ *belonging to the set* $V$, *there exists* $\varepsilon' > 0$ *and* $\delta' \in (0, 1)$ *such that:*

$$\textit{convex hull}\,((x + U_{X_2}(\varepsilon', \delta')) \cap K(U, V)) \subseteq x + U_{X_2}(\varepsilon, \delta).$$

*In that case, there is at least one pair* $(u, v) \in U \times V$ *satisfying the following system of equations:*

$$u = H(u, v), \quad v = K(u, v).$$

*Proof.* Following Theorem 5.8, we can conclude that the function $\eta : v \to \eta v$, defined as $\eta v = H(\eta v, v)$ for $v \in V$, is continuous. Now, if we define the function $G : V \to V$ as $Gv = K(\eta v, v)$ for $v \in V$, then the function $G$ is compact and satisfies all the conditions necessary for Rzepecki's fixed point theorem. Consequently, we can affirm that $\Phi(G) \neq \emptyset$. For any $v \in \Phi(G)$, we have $u = \eta v$. This completes the proof. ■

## 6. Conclusion and Future Work

This paper has focused on the study of fuzzy quasi-metric spaces, particularly in the context of identifying fixed points for specific mappings within this framework. We have developed tailored fixed-point theorems that accommodate various types of mappings, ensuring their practical relevance in applications. Additionally, akin to Kransnoselski's theorems, we have established analogous rules applicable to these spaces, further enriching their theoretical foundation. Moreover, our research has explored the stability of fixed points under specific conditions, providing insights into the robustness of our findings. To elucidate these theoretical advancements, we have included illustrative examples that demonstrate the practical implications of our results. In conclusion, this paper contributes to the ongoing developments in the field of fuzzy quasi-metric spaces by presenting novel fixed-point theorems and exploring their applications.

Future research directions emerging from this paper include extending the applicability of the developed fixed-point theorems to broader fuzzy quasi-metric spaces and related structures such as fuzzy metric spaces or probabilistic metric spaces. Further investigations will explore the stability of fixed points under varied conditions and assess their robustness to perturbations in mappings or the space itself. Practical applications in fields like computer science, economics, and engineering will be pursued, leveraging the developed theories to address specific problems. Comparative studies will evaluate the effectiveness and scope of the fixed-point theorems against alternative methodologies in solving problems within fuzzy quasi-metric spaces, highlighting their respective strengths and limitations. Additionally, exploration of novel properties of fixed points beyond stability, such as convergence rates and uniqueness under specific conditions, will be undertaken. Empirical validation through computational experiments will substantiate theoretical findings, affirming the reliability and applicability of the developed theorems. Furthermore, efforts will focus on creating educational resources that clarify the concepts and practical uses of fixed-point theorems in fuzzy quasi-metric spaces, catering to both academic and practical audiences. These endeavors aim to build upon the foundational contributions of this paper, fostering continued advancements in the field.

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