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A study on Fibonacci-Euler sequence spaces and related matrix transformations

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Abstract. In this paper, we study the domains $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$ of the matrix, involving Fibonacci and Lucas numbers, in the spaces c_0 and c, respectively. Apart from some basic topological properties, we give the Schauder basis for them. As well, we determine the $\alpha-$, $\beta-$, $\gamma-$duals and characterize certain matrix classes related to these spaces.

1. Introduction and preliminaries

As is well known, the Fibonacci sequence (F_n) was defined by Leonardo Fibonacci as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. This sequence has many interesting and meaningful properties, and applications such as the golden ratio in arts, sciences and architecture. Similar to the Fibonacci numbers, the Lucas numbers can be defined $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$.

There are many elegant relations concerning Fibonacci numbers. Two of them are as follows [37]:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1, \quad \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \text{ for each } n \in \mathbb{N}$$

meanwhile analogous results for the Lucas family are valid as

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3, \quad \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2, \text{ for each } n \in \mathbb{N}.$$

Numerous computational and summation formulas, including both of them were recorded in the literature [37, Vol 1, p. 251]. For instance, we may give the relation

$$\sum_{k=0}^{n} \binom{n}{k} F_k L_{n-k} = 2^n F_n, \tag{1}$$

which is the main motivation of this paper.

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1.1. Sequence spaces

Let the space of all real-valued sequences be denoted by ω and recall that any vector subspace of ω is a sequence space. One can give the celebrated sequence spaces by $\ell_p, \ell_\infty, c, c_0$ as the set of p-absolutely summable, bounded, convergent, and null sequences, respectively.

Let X be a Banach space. Then, it is called as a BK-space if each map $p_k : X \to \mathbb{R}$ defined by $p_k(z) = z_k$ is continuous for all $k \in \mathbb{N}$ whereas a complete linear metric space is said to be a FK-space with continuous coordinate functionals.

One can consider an infinite matrix as a linear operator from a sequence space to another one. Let $T = (t_{nk})$ be an infinite matrix with real or complex entries and let X and Y be two sequence spaces. Then, T becomes a matrix transformation $X \to Y$ if for each sequence $z = (z_k) \in X$, $Tz \in Y$, i.e., the $T-$transform of z, where $(Tz)_n = \sum_k t_{nk} z_k$ for $n \in \mathbb{N}$. By (X, Y), we denote the class of all matrices T such that $T : X \to Y$.

In this study, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{R}$ denotes the set of all real numbers. For simplicity in notation, in the sequel, the summation without limits runs from 0 to ∞.

The matrix domain of an infinite matrix T in X, denoted by X_T, is defined by

$$X_T = \{z \in \omega : Tz \in X\}.$$

For details concerning the domain of special matrices in normed sequence spaces, the reader can investigate the monograph of Başar [9].

1.2. Motivation and aim

Creating new sequence spaces using a special limitation method with the help of matrix domain and researching their topological, algebraic features and matrix transformations have been intensively studied. Especially, Fibonacci and Lucas sequence have been applied so as to introduce sequence spaces and cope with their poperties. We may refer the reader to the studies [7, 12, 17, 18, 24, 27–29, 31, 33, 34]. for the developments in Fibonacci and Lucas sequence spaces' direction and [2, 3, 13–16, 19, 21, 22, 25, 30, 32, 36, 40, 45, 46] on some new sequence spaces generated by certain triangle matrices. The reader can refer to the monograph [38] and references therein, devoted to the new developments for summability theory and related topics, and recent papers [4, 8, 11, 43] concerning the domains of certain triangles in some classical sequence spaces.

Indeed, the sequence space theory contains a useful tool for acquiring the geometrical and topological results through the Schauder basis. By adopting this fact, construction of the Euler sequence spaces via Euler matrix and its some generalizations has been studied and lots of remarkable conclusions have been revealed in the literature, (see for instance [1, 5, 6, 10, 23, 35, 39, 41]).

Quite recently, the authors of [20] introduced the following regular matrix $F^{L,E} = \left(f_{nk}^{L,E}\right)$, defined by

$$f_{nk}^{L,E} = \begin{cases} \binom{n}{k} 2^{-n} \dfrac{F_k L_{n-k}}{F_n}, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n; \end{cases} \tag{2}$$

motivated by (1). Also, the domain of this regular matrix in the spaces ℓ_p and ℓ_∞ is studied and certain topological features containing Schauder basis and the $\alpha-$dual, $\beta-$dual and $\gamma-$duals and some matrix characterizations are achieved.

As a natural continuation of [20], we intend to develop the domains $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$ of the matrix above including Fibonacci and Lucas numbers in the spaces c_0 and c, respectively. Some topological properties are exhibited and the Schauder basis is given for them. Furthermore, the $\alpha-$, $\beta-$, $\gamma-$duals of the spaces and certain matrix classes are established.

It is worth mentioning that the matrix $F^{L,E}$ defined by (2) can be regarded as the composition of the matrices Euler matrix E_1 of order 1 and Fibonacci-Lucas matrix, that can lead to further interesting and meaningful results in this concept.

2. The sequence spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$

Let us introduce the following sequence spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$ as the set of all sequences whose $F^{L,E}-$transforms (see (3) below) are in c_0 and c, respectively. Namely,

$$c_0\left(F^{L,E}\right)=\left\{z=(z_k)\in\omega:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}2^{-n}\frac{F_kL_{n-k}}{F_n}z_k=0\right\}$$

and

$$c\left(F^{L,E}\right)=\left\{z=(z_k)\in\omega:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}2^{-n}\frac{F_kL_{n-k}}{F_n}z_k \text{ exists}\right\}.$$

We also give the following sequence $y=(y_n)$ defined by $F^{L,E}-$transform of a sequence $z=(z_k)$

$$y_n=\left(F^{L,E}z\right)_n=\sum_{k=0}^{n}\binom{n}{k}2^{-n}\frac{F_kL_{n-k}}{F_n}z_k, \tag{3}$$

for all $n\in\mathbb{N}$.

It should be stated that the spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$ are BK-spaces endowed with the norms

$$\|z\|_{c_0(F^{L,E})}=\|z\|_{c(F^{L,E})}=\left\|F^{L,E}z\right\|_{\ell_\infty}=\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\binom{n}{k}2^{-n}\frac{F_kL_{n-k}}{F_n}z_k\right|,$$

can be reached by Wilansky [44, Theorem 4.3.2].

Theorem 2.1. *The spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$ are linearly isomorphic to c_0 and c, respectively.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0\left(F^{L,E}\right)$ and c_0. The linearity is clear. It is satisfied $z=0$ whenever $F^{L,E}z=0$ yields the injectiveness of $F^{L,E}$. Consider a sequence $y=(y_n)\in c_0$, if the sequence $z=(z_k)$ is denoted by

$$z_k=\sum_{i=0}^{k}\left((-1)^{k-i}\binom{k}{i}2^{i-1}\frac{F_i}{F_k}P_{k-i}\right)y_i \text{ for } k\in\mathbb{N} \tag{4}$$

where P_k is the determinant

$$\begin{vmatrix} L_1 & L_0 & 0 & \dots & 0\\ L_2 & L_1 & L_0 & \dots & 0\\ L_3 & L_2 & L_1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ L_k & L_{k-1} & L_{k-2} & \dots & L_1 \end{vmatrix}$$

for all $k\in\mathbb{N}\setminus\{0\}$ subject to initial condition $P_0=1$, then,

$$\begin{aligned}\left(F^{L,E}z\right)_n&=\frac{1}{2^nF_n}\sum_{k=0}^{n}\binom{n}{k}F_kL_{n-k}z_k\\ &=\frac{1}{2^nF_n}\sum_{k=0}^{n}\binom{n}{k}F_kL_{n-k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}2^{i-1}\frac{F_i}{F_k}P_{k-i}y_i\\ &=y_n.\end{aligned}$$

Thus, for $z=(z_k)$ given by (4) as $y=(y_n)\in c_0$, one can see that $\left(F^{L,E}z\right)_n=y_n$ for all $n\in\mathbb{N}$. By $F^{L,E}z=y$, the mapping $F^{L,E}$ is onto. $F^{L,E}$ is norm-preserving from $\|z\|_{c_0(F^{L,E})}=\|y\|_{\ell_\infty}$. Consequently, $c_0\left(F^{L,E}\right)$ and c_0 are linearly isomorphic. To derive the other case of the theorem, replace $c_0\left(F^{L,E}\right)$ and c_0 by $c\left(F^{L,E}\right)$ and c, respectively. This finishes the proof. □

Now, for the purpose of the construction of the Schauder basis for the domain of the matrix, we give the definition a Schauder basis of a normed space. If a normed space $(\lambda, \|.\|)$ contains a sequence (δ_n) such that for every $x \in \lambda$, there exists a unique sequence of scalars (τ_n) for which

$$\lim_{n\to\infty}\left\|x-\sum_{k=0}^{n}\tau_k\delta_k\right\|=0,$$

then, we say that (δ_n) is a Schauder basis for λ, and we write

$$x=\sum_{k=0}^{\infty}\tau_k\delta_k.$$

Combining the fact that the domain X_T of an infinite matrix T in X has a basis if and only if X has a basis, and Theorem 2.1 allows us to present the following theorem.

Theorem 2.2. *Let $\psi^{(k)} \in c_0\left(F^{L,E}\right)$ for each $k \in \mathbb{N}$, and let the sequence $\psi^{(k)} = \left\{\psi_n^{(k)}\right\}_{n\in\mathbb{N}}$ be defined by*

$$\psi_n^{(k)}=\begin{cases}(-1)^{n-k}\binom{n}{k}2^{k-1}\frac{F_k}{F_n}P_{n-k}, & \text{if } 0\le k\le n;\\ 0, & \text{if } k>n.\end{cases}.$$

Then, (1): The set $\left\{\psi^{(0)},\psi^{(1)},...\right\}$ is a basis for the space $c_0\left(F^{L,E}\right)$ and any z in $c_0\left(F^{L,E}\right)$ is uniquely determined as $z=\sum_k t_k\psi^{(k)}$. (2): For $\mu=\lim_{k\to\infty}\left(F^{L,E}z\right)_k$, the set $\left\{e,\psi^{(0)},\psi^{(1)},...\right\}$ is a basis for the space $c\left(F^{L,E}\right)$ and any z in $c\left(F^{L,E}\right)$ is uniquely determined as $z=\mu e+\sum_k\left(t_k-\mu\right)\psi^{(k)}$.

3. $\alpha-$dual, $\beta-$dual and $\gamma-$duals

We devote this section to determining the $\alpha-$dual, $\beta-$dual and $\gamma-$dual of the spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$.

First, we recall the definition of duals of the spaces. By $S(X,Y)$, we denote the multiplier space of X and Y, defined by

$$S(X,Y)=\{u\in\omega : zu\in Y \text{ for all } z\in X\}.$$

In this context, if cs and bs represent the spaces of sequences with convergent and bounded series, respectively, then, the $\alpha-$dual, $\beta-$dual and $\gamma-$dual of a sequence space X can be given as

$$X^{\alpha}=S(X,\ell_1),\;\; X^{\beta}=S(X,cs) \text{ and } X^{\gamma}=S(X,bs).$$

Lemma 3.1. *([42]) $T=(t_{nk})\in(c_0,\ell_1)=(c,\ell_1)$ if and only if*

$$\sup_{N,M\in F}\left|\sum_{n\in N}\sum_{k\in M}t_{nk}\right|<\infty,$$

where F denotes the family of all finite subsets of $\mathbb{N}$.

Lemma 3.2. *([3, Theorem 3.1]) For any triangular matrix $\mathcal{U}=(u_{nk})$ with the inverse $\mathcal{V}=(v_{nk})$, if the matrix $\mathcal{P}=(p_{nk})$ is as follows*

$$p_{nk}=\begin{cases}\sum_{j=k}^{n}b_jv_{jk}, & \text{if } 0\le k\le n;\\ 0, & \text{if } k>n;\end{cases}$$

then,

$$X_U^\beta = \{b = b_k \in \omega : \mathcal{P} \in (X, c)\}$$

and

$$X_U^\gamma = \{b = b_k \in \omega : \mathcal{P} \in (X, \ell_\infty)\}.$$

Our theorems are now in order.

Theorem 3.3. *For the spaces $c_0\left(F^{L,E}\right)$ and $c\left(F^{L,E}\right)$, the α−dual is as in the following:*

$$\varphi_1 = \left\{ b = (b_n) \in \omega : \sup_{N,M \in F} \left| \sum_{n \in N} \sum_{k \in M} (-1)^{n-k} \binom{n}{k} 2^{k-1} \frac{F_k}{F_n} P_{n-k} b_n \right| < \infty \right\}.$$

Proof. Define the matrix $\mathcal{S} = (s_{nk})$ by the relation in terms of any sequence $b = (b_n) \in \omega$:

$$s_{nk} = \begin{cases} (-1)^{n-k} \binom{n}{k} 2^{k-1} \frac{F_k}{F_n} P_{n-k} b_n, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

Apply (3) to get for all $n \in \mathbb{N}$ that

$$(\mathcal{S}y)_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^{k-1} \frac{F_k}{F_n} P_{n-k} b_n y_k = b_n z_n,$$

from which $bz \in \ell_1$ for $z \in c_0\left(F^{L,E}\right)$ or $c\left(F^{L,E}\right)$ if and only if $\mathcal{S}y \in \ell_1$ for $y \in c_0$ or $y \in c$. Hence, use Lemma 3.1 to arrive at

$$\sup_{N,M \in F} \left| \sum_{n \in N} \sum_{k \in M} (-1)^{n-k} \binom{n}{k} 2^{k-1} \frac{F_k}{F_n} P_{n-k} b_n \right|,$$

and consequently, $\left(c_0\left(F^{L,E}\right)\right)^\alpha = \left(c\left(F^{L,E}\right)\right)^\alpha = \mathcal{S}$. This finishes the proof. □

Theorem 3.4. *Define the sets φ_2, φ_3 and φ_4 by*

$$\varphi_2 = \left\{ b = (b_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} 2^{k-1} \frac{F_k}{F_i} P_{i-k} b_i \right| < \infty \right\},$$

$$\varphi_3 = \left\{ b = (b_k) \in \omega : \lim_{n \to \infty} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} 2^{k-1} \frac{F_k}{F_i} P_{i-k} b_i \text{ exists for each } k \in \mathbb{N} \right\}$$

and

$$\varphi_4 = \left\{ b = (b_k) \in \omega : \lim_{n \to \infty} \sum_{k} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} 2^{k-1} \frac{F_k}{F_i} P_{i-k} b_i \text{ exists} \right\}.$$

Then, $\left(c_0\left(F^{L,E}\right)\right)^\beta = \varphi_2 \cap \varphi_3$, $\left(c\left(F^{L,E}\right)\right)^\beta = \varphi_2 \cap \varphi_3 \cap \varphi_4$, $\left(c_0\left(F^{L,E}\right)\right)^\gamma = \left(c\left(F^{L,E}\right)\right)^\gamma = \varphi_2$.

Proof. The proof follows from Lemma 3.2 by using similar arguments to Theorem 3.3. So, we omit it. □

4. Certain matrix mappings

The aim of this section is to establishing the characterization of certain classes on the space $\left(X\left(F^{L,E}\right), Y\right)$ for $X \in \{c_0, c\}$ and $Y = \{\ell_1, c_0, c, \ell_\infty\}$. We require the following lemmas, given by [42].

Lemma 4.1. *$T = (t_{nk}) \in (c, c)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}| < \infty \tag{5}$$

holds and there exists $\lambda_k \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\lim_{n\to\infty} t_{nk} = \lambda_k \tag{6}$$

for each $k \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sum_k t_{nk} = \lambda.$$

Also, $T = (t_{nk}) \in (c_0, c_0)$ if and only if (6) holds and

$$\lim_{n\to\infty} t_{nk} = 0 \tag{7}$$

for each $k \in \mathbb{N}$.

Lemma 4.2. *$T = (t_{nk}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ if and only if (5) is satisfied. In addition, $T = (t_{nk}) \in (c_0, c)$ if and only if the relations (5) and (6) hold.*

Lemma 4.3. *$T = (t_{nk}) \in (c, c_0)$ if and only if (5) and (7) are satisfied and $\lim_{n\to\infty} \sum_k t_{nk} = 0$.*

We also need the following Theorem 4.1 of [26].

Theorem 4.4. *Let X be a FK-space, $\mathcal{V} = (v_{nk})$ be the inverse matrix of the triangle matrix $\mathcal{U} = (u_{nk})$ and Y be an arbitrary subset of ω. Then, $T = (t_{nk}) \in (X_{\mathcal{U}}, Y)$ if and only if*

$$G^{(n)} = \left(g^{(n)}_{mk}\right) \in (X, c), \quad \textit{for each } n \in \mathbb{N},$$

and

$$G = (g_{nk}) \in (X, Y),$$

where

$$g^{(n)}_{mk} = \begin{cases} \sum_{i=k}^{m} t_{ni} v_{ik}, & \textit{if } 0 \le k \le m; \\ 0, & \textit{if } k > m, \end{cases}$$

and

$$g_{nk} = \sum_{i=k}^{\infty} t_{ni} v_{ik}, \quad \textit{for all } k, m, n \in \mathbb{N}.$$

Now, we list some conditions:

$$\sup_{j\in\mathbb{N}} \sum_{k=0}^{j} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{i}{k} 2^{k-1} \frac{F_k}{F_i} P_{i-k} t_{ni} \right| < \infty \text{ for each fixed } n \in \mathbb{N}. \tag{8}$$

$$\lim_{j\to\infty}\sum_{i=k}^{j}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni} \text{ exists for each fixed } k,n\in\mathbb{N}. \tag{9}$$

$$\lim_{j\to\infty}\sum_{k=0}^{j}\sum_{i=k}^{j}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni} \text{ exists for each fixed } n\in\mathbb{N}. \tag{10}$$

$$\sup_{N,M\in F}\left|\sum_{n\in N}\sum_{k\in M}\sum_{i=k}^{\infty}(-1)^{n-k}\binom{n}{k}2^{k-1}\frac{F_k}{F_n}P_{n-k}t_{ni}\right|<\infty. \tag{11}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni}\right|<\infty. \tag{12}$$

$$\lim_{n\to\infty}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni}=0 \text{ for each } k\in\mathbb{N}. \tag{13}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni}=0. \tag{14}$$

$$\lim_{n\to\infty}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni} \text{ exists for each } k\in\mathbb{N}. \tag{15}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ni} \text{ exists.} \tag{16}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{r=0}^{n}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ri}\right|<\infty. \tag{17}$$

$$\lim_{n\to\infty}\sum_{r=0}^{n}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ri}=0 \text{ for each } k\in\mathbb{N}. \tag{18}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{r=0}^{n}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ri}=0. \tag{19}$$

$$\lim_{n\to\infty}\sum_{r=0}^{n}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ri} \text{ exists for each } k\in\mathbb{N}. \tag{20}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{r=0}^{n}\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{k}2^{k-1}\frac{F_k}{F_i}P_{i-k}t_{ri} \text{ exists.} \tag{21}$$

We are in position to mention our theorem.

Theorem 4.5. *Let* $X\in\left\{c_0\left(F^{L,E}\right),c\left(F^{L,E}\right)\right\}$ *and* $Y\in\{\ell_1,c_0,c,\ell_\infty\}$. *Then, the characterization for* $T=(t_{nk})\in(X,Y)$ *can be observed in Table 1.*

1. (8), (9) and (11) hold.
2. (8), (9), (10) and (11) hold.
3. (8), (9), (12) and (13) hold.
4. (8), (9), (10), (12) (13) and (14) hold.
5. (8), (9), (12) and (15) hold.
6. (8), (9), (10), (12), (15) and (16) hold.
7. (8), (9) and (12) hold.
8. (8), (9), (10) and (12) hold.

Corollary 4.6. *Let* $X\in\left\{c_0\left(F^{L,E}\right),c\left(F^{L,E}\right)\right\}$ *and* $Y\in\{cs_0,cs,bs\}$. *Then, the characterization for* $T=(t_{nk})\in(X,Y)$ *can be observed in Table 2.*

1. (8), (9), (17) and (18) hold.
2. (8), (9), (10), (17), (18) and (19) hold.
3. (8), (9), (17) and (20) hold.
4. (8), (9), (10), (17) (20) and (21) hold.
5. (8), (9) and (17) hold.
6. (8), (9), (10) and (17) hold.

Table 1: The characterization

	To			
From	l_1	c_0	c	l_∞
$c_0\left(F^{L,E}\right)$	1	3	5	7
$c\left(F^{L,E}\right)$	2	4	6	8

Table 2: The characterization

	To		
From	l_1	c_0	c
$c_0\left(F^{L,E}\right)$	1	3	5
$c\left(F^{L,E}\right)$	2	4	6

Lemma 4.7. *([42])$T = (t_{nk}) \in (\ell_1, c_0)$ if and only if (7) holds and*

$$\sup_{n,k\in\mathbb{N}} |t_{nk}| < \infty. \tag{22}$$

$T = (t_{nk}) \in (\ell_\infty, c_0)$ if and only if (7) holds and

$$\lim_{n\to\infty} \sum_k |t_{nk}| = 0.$$

$T = (t_{nk}) \in (\ell_1, c)$ if and only if (6) and (22) hold. Also, $T = (t_{nk}) \in (\ell_\infty, c)$ iff (6) holds and

$$\lim_{n\to\infty} \sum_k |t_{nk}| = \sum_k \left|\lim_{n\to\infty} t_{nk}\right|.$$

Theorem 4.8. *Let $\mathcal{A} = (a_{nk})$ and $\mathcal{B} = (b_{nk})$ be infinite matrices, whose entries have relationship as follows:*

$$b_{nk} = \sum_{i=0}^{n} \binom{n}{i} 2^{-n} \frac{F_i L_{n-i}}{F_n} a_{ik},$$

for all $n, k \in \mathbb{N}$. Then, $\mathcal{A} \in \left(Y, X\left(F^{L,E}\right)\right)$ if and only if $\mathcal{B} \in (Y, X)$ for all $X = \{c_0, c\}$ and any sequence space Y.

Proof. For any $z = (z_k) \in Y$, we can write

$$\sum_{k=0}^{\infty} b_{nk} z_k = \sum_{i=0}^{n} \binom{n}{i} 2^{-n} \frac{F_i L_{n-i}}{F_n} \sum_{k=0}^{\infty} a_{ik} z_k,$$

from which $(\mathcal{B}z)_n = \left(F^{L,E}(\mathcal{A}z)\right)_n$ for all $n \in \mathbb{N}$. So, $\mathcal{A}z \in X\left(F^{L,E}\right)$ for $z = (z_k) \in Y$ if and only if $\mathcal{B}z \in X$ for $z = (z_k) \in Y$. As a result, the proof is complete. □

The followings are some further conditions:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik} = 0 \text{ for each } k\in\mathbb{N}. \tag{23}$$

$$\sup_{k,n\in\mathbb{N}}\left|\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik}\right| < \infty. \tag{24}$$

$$\lim_{n\to\infty}\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik} \text{ exists for each } k\in\mathbb{N}. \tag{25}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik}\right| < \infty. \tag{26}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik} = 0. \tag{27}$$

$$\lim_{n\to\infty}\sum_{k}\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik} \text{ exists}. \tag{28}$$

$$\lim_{n\to\infty}\sum_{k}\left|\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik}\right| = 0. \tag{29}$$

$$\lim_{n\to\infty}\sum_{k}\left|\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik}\right| = \sum_{k}\left|\lim_{n\to\infty}\sum_{i=0}^{n}\binom{n}{i}2^{-n}\frac{F_iL_{n-i}}{F_n}t_{ik}\right|. \tag{30}$$

Theorem 4.9. *Let $X\in\left\{c_0\left(F^{L,E}\right), c\left(F^{L,E}\right)\right\}$ and $Y\in\{\ell_1, c_0, c, \ell_\infty\}$. Then, the characterization for $T=(t_{nk})\in(Y,X)$ can be observed in Table 3.*

1. (23) and (24) hold.
2. (24) and (25) hold.
3. (23) and (26) hold.
4. (25) and (26) hold.
5. (23), (26) and (27) hold.
6. (25), (26) and (28) hold.
7. (23) and (29) hold.
8. (25) and (30) hold.

Table 3: The characterization

	To	
From	$c_0\left(F^{L,E}\right)$	$c\left(F^{L,E}\right)$
l_1	1	2
c_0	3	4
c	5	6
l_∞	7	8

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