Filomat 39:9 (2025), 2985–3001
https://doi.org/10.2298/FIL2509985E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The group of autohomeomorphisms of some digital topologies on the integers

Othman Echi[a]

[a]*Department of Mathematics, College of Computing and Mathematic, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*

Abstract. For a topological space X, the group of autohomeomorphisms is denoted by $\mathcal{H}(X)$. It is a well-established fact that even if two topological spaces X and Y have isomorphic autohomeomorphism groups, it does not necessarily imply that X and Y are homeomorphic. A space X is considered homogeneous if its autohomeomorphism group, $\mathcal{H}(X)$, acts transitively on X, via the action

$$\begin{array}{ccc} \mathcal{H}(X) \times X & \longrightarrow & X \\ (g, x) & \longmapsto & g(x). \end{array}$$

The degree of homogeneity of X, denoted as $d_H(X)$, is defined as the cardinality of the quotient set $X/\mathcal{H}(X)$ relative to the aforementioned action.

Regarding the Khalimsky topology defined on the set of integers, this topology, denoted by $\mathcal{K}$, is the topology generated by the family

$$\{\{x-1, x, x+1\} : x \text{ is an even integer}\},$$

as a subbase. The space $(\mathbb{Z}, \mathcal{K})$, known as the *Khalimsky line* or *digital line*, will be denoted by **KL** (or $\mathbf{KL}_1$). The digital line is notably influential in digital image processing and computer graphics. For recent advancements in digital topologies on $\mathbb{Z}^n$, see [3], [10], [11], and [12].

The aim of this paper is the construction of a sequence of Alexandroff topologies, $\{\mathcal{K}_p : p \in \mathbf{N}\}$, on the set of integers $\mathbb{Z}$. This provides new digital topologies with the following properties:

- $\mathcal{H}(\mathbb{Z}, \mathcal{K}_p)$ is isomorphic to $\mathcal{H}(\mathbb{Z}, \mathcal{K})$.
- For each positive integer p, $(\mathbb{Z}, \mathcal{K}_p)$ is topologically embedded in $(\mathbb{Z}, \mathcal{K}_{p+1})$.

The degree of homogeneity, Krull dimension, inductive dimension and the height of $(\mathbb{Z}, \mathcal{K}_p)$ are also computed.

1. Introduction

For a topological space X, we let $\mathcal{H}(X)$ denote the group of autohomeomorphisms of X. A classic query, originating from Ulam [27], inquires if spaces X and Y are homeomorphic when their autohomeomorphism

2020 *Mathematics Subject Classification*. Primary 22F50; Secondary 20B27, 54F65
Keywords. Alexandroff space, degree of homogeneity, group of autohomeomorphisms, Khalimsky line, reversible space
Received: 9 February 2024; Revised: 20 November 2024; Accepted: 24 November 2024
Communicated by Ljubiša D. R. Kočinac
Research supported by DSR, King Fahd University of Petroleum and Minerals.
Email address: othechi@yahoo.com (Othman Echi)
ORCID iD: https://orcid.org/0000-0002-0082-7644 (Othman Echi)

groups $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are algebraically isomorphic. Although simple examples demonstrate that the answer is generally no, there are specific instances where the answer is yes. For example, as detailed by Whittaker [28], in the context of compact manifolds, if there exists a group isomorphism $\varphi : \mathcal{H}(X) \longrightarrow \mathcal{H}(Y)$, then there exists a homeomorphism $\omega : X \longrightarrow Y$ such that $\varphi(h) = \omega h \omega^{-1}$, for all $h \in \mathcal{H}(X)$.

A space X is said to be homogeneous if $\mathcal{H}(X)$ acts transitively on X by the action

$$\begin{array}{ccc} \mathcal{H}(X) \times X & \longrightarrow & X \\ (g, x) & \longmapsto & g(x). \end{array}$$

The *degree of homogeneity* of X, denoted by $d_H(X)$, is defined by the cardinality of the quotient set $X/\mathcal{H}(X)$ according to the aforementioned action. When $d_H(X) = n$, X is referred to as $\frac{1}{n}$-homogeneous.

It is noteworthy that the authors of [21] provided significant insights for the autohomeomorphism group of certain typical Alexandroff spaces.

The main goal of this paper is to develop new Alexandroff topologies, denoted by $\mathcal{K}_p$, on $\mathbb{Z}$ such that $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{H}(\mathcal{K})$ is identical to $\mathcal{H}(\mathcal{K}_p)$, for each integer p, and that each topology $\mathcal{K}_p$ is embedded within $\mathcal{K}_{p+1}$.

Let us review some fundamental concepts that will be utilized consistently in this paper.

An *adjacency relation* on a non-empty set V is defined as a symmetric and irreflexive binary relation π. If for any two elements x and y in V, there exists a finite sequence of elements $x_0, \ldots, x_n$ from V such that $x = x_0$, $y = x_n$, and $(x_j, x_{j+1}) \in \pi$ for $j \in \{0, 1, \ldots, n-1\}$, then the pair (V, π) is called a *digital space*, as described by Herman [13]. In this context, if $(x, y) \in \pi$, then x and y are considered to be *π-connected*.

In a topological space X, two distinct points x and y are said to be *adjacent* when the subspace consisting of $\{x, y\}$ is connected. The set of all points y that are adjacent to x, denoted by $\mathcal{A}_X(x)$, is called the *adjacency set* of x. This specific type of adjacency relation is known as the *topological adjacency* in the space X.

Points x and y in a space are adjacent precisely when $x \neq y$, and either $y \in \mathcal{N}(x)$ or $x \in \mathcal{N}(y)$, where $\mathcal{N}(x)$ denotes the intersection of all open sets in X that contain x. An alternative equivalent condition is that $x \neq y$, and either $y \in \overline{\{x\}}$ or $x \in \overline{\{y\}}$. Therefore, the adjacency set of x in X, $\mathcal{A}_X(x)$, can be expressed as:

$$\mathcal{A}_X(x) = (\mathcal{N}(x) \cup \overline{\{x\}}) \setminus \{x\}.$$

An Alexandroff topology is a topology where the intersection of any family of open sets is also open. This kind of topologies were initially introduced and explored by Alexandroff in 1937 [1], under the designation *"Diskrete Räume"*. In Steiner's work [26], Alexandroff spaces were studied under the term *"principal spaces"*." Steiner demonstrated that in the lattice of topologies on any set, each topology possesses a principal topology complement.

Alexandroff spaces are known for their versatile properties, finding applications in various fields such as the "geometry of the computer screen" and digital topology. Alexandroff topological spaces with the T_0-separation axiom have connections with partial orders. For recent developments in Alexandroff spaces and insights into how topology and ordered sets are applied in information theory, one can refer to: [16], [2] and [20].

Consider a preorder (i.e. a reflexive and transitive binary relation) $\leq$ on a set X. For an element x in X, we denote :

- the set $\{y \in X : y \leq x\}$ by $(\downarrow x]$, and
- the set $\{y \in X : x \leq y\}$ by $[x \uparrow)$.

The topology on X, with base $\mathcal{B} = \{(\downarrow x] : x \in X\}$, is denoted by $\mathcal{A}(\leq)$. It is clear that $\mathcal{A}(\leq)$ constitutes an Alexandroff topology. This topology, $\mathcal{A}(\leq)$, is termed as the *Alexandroff topology induced by the preorder* $\leq$. In this topology, the closure of a singleton set $\{x\}$ is given by $[x \uparrow)$ for every element x in X.

In a topological space X, the preorder $\leq_{\mathcal{T}}$ defined by the condition

$$x \leq_{\mathcal{T}} y \text{ if and only if } y \in \overline{\{x\}}$$

is known as the *preorder induced by the topology* of X; it is also called the *opposite specialization order*.

If $(X, \mathcal{T})$ is an Alexandroff space and $\leq_{\mathcal{T}}$ is the preorder induced by $\mathcal{T}$, then $\mathcal{T} = \mathcal{A}(\leq_{\mathcal{T}})$.

Let **Alex** be the category with objects the Alexandroff spaces and with arrows the continuous maps. We let also **Pord** be then the category with objects the preordered sets and with arrows the preorder preserving maps. Then **Alex** and **Pord** are isomorphic.

It is a well-established fact that if π is the topological adjacency on an Alexandroff space X, then (X, π) constitutes a digital space if and only if X is π-connected as per [13, Theorem 4.2.2.].

The digital analog of the Euclidean topology on the real line, invented by Efim Khalimsky, features notably in this context (refer to [18], [17] and [19], for more information).

The *Khalimsky* line refers to the set of integers $\mathbb{Z}$ equipped with the topology $\mathcal{K}$ such that a subset U of $\mathbb{Z}$ is $\mathcal{K}$-open if and only if whenever $2n \in U$, then $2n-1, 2n+1 \in U$. It follows that the Khalimsky topology is Alexandroff and for every $n \in \mathbb{Z}$, the following properties hold.

- If n is odd, then $\{n\}$ is $\mathcal{K}$-open, $\mathcal{N}(n) = (\downarrow n] = \{n\}$, and $\overline{\{n\}} = \{n-1, n, n+1\} = [n \uparrow)$.
- If n is even, then $\{n\}$ is $\mathcal{K}$-closed, $\mathcal{N}(n) = (\downarrow n] = \{n-1, n, n+1\}$, and $\overline{\{n\}} = \{n\} = [n \uparrow)$.

Defining $\leq_1$ as the ordering induced by the Khalimsky topology $\mathcal{K}$, the relation $x \leq_1 y$ holds true if and only if either $x = y$ or y is even and $x \in \{y-1, y+1\}$. In this context, the partially ordered set (poset) $(\mathbb{Z}, \leq_1)$ can be visualized as follows:

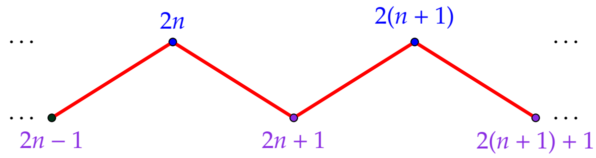

Figure 1: The poset $(\mathbb{Z}, \leq_1)$.

The Khalimsky line, represented by the set of integers $\mathbb{Z}$ equipped with the Khalimsky topology $\mathcal{K}$, will be denoted by **KL**.

A point x of a connected topological space X is called a cut-point if the subspace $X \setminus \{x\}$ is disconnected, A *cut-point space* is defined as a connected space in which every point is a cut-point. This space is considered a *minimal cut-point space* if none of its proper subspaces qualifies as a cut-point space. According to [15], there exists only one unique minimal cut-point space, when considered up to homeomorphism, namely the Khalimsky line.

For $\{a, b\} \subset \mathbb{Z}$ with $a \leq b$, the notation $[a, b]_{\mathbb{Z}}$ is considered as $\{t \in \mathbb{Z} : a \leq t \leq b\}$. The set $\{t \in \mathbb{Z} : a \leq t < b\}$ will be denoted by $[a, b)_{\mathbb{Z}}$.

In the subsequent definition, we will introduce the concept of the p-Khalimsky topology.

Definition 1.1. *Given a positive integer p, the p-Khalimsky topology on $\mathbb{Z}$, denoted by $\mathcal{K}_p$, is the topology generated by the union of the following collections of sets:*

$$\{[2np - p, 2np + p]_{\mathbb{Z}} : n \in \mathbb{Z}\},$$

$$\bigcup_{n \in \mathbb{Z}} \{[2np - p, 2np - i]_{\mathbb{Z}} : i \in [1, p]_{\mathbb{Z}}\},$$

$$\bigcup_{n \in \mathbb{Z}} \{[2np + i, 2np + p]_{\mathbb{Z}} : i \in [1, p]_{\mathbb{Z}}\},$$

as a base.

It is clear that a set $O \subseteq \mathbb{Z}$ is $\mathcal{K}_p$-open if and only if for every $i \in [0, p]_{\mathbb{Z}}$ and for every integer $x \in 2p\mathbb{Z}$ (the set of multiples of $2p$ in $\mathbb{Z}$), the set O satisfies the following properties.

1. If $x - i \in O$, then $[x - p, x - i]_{\mathbb{Z}} \subseteq O$.
2. If $x + i \in O$, then $[x + i, x + p]_{\mathbb{Z}} \subseteq O$.

The p-Khalimsky topology $\mathcal{K}_p$, when specified for $p = 1$, aligns with the original Khalimsky topology, denoted as $\mathcal{K}$. This means that $\mathcal{K}_1$ coincides with the Khalimsky topology $\mathcal{K}$.

In the forthcoming sections, we will show that the following properties hold.

- The group of homeomorphisms $\mathcal{H}(\mathbf{KL}_p)$ is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$.
- The degree of homogeneity $d_H(\mathbf{KL}_p)$ for each $\mathbf{KL}_p$ is equal to $p + 1$.
- There exists an increasing sequence

$$\mathbf{KL}_1 \overset{f_1}{\hookrightarrow} \mathbf{KL}_2 \hookrightarrow \dots \hookrightarrow \mathbf{KL}_p \overset{f_p}{\hookrightarrow} \mathbf{KL}_{p+1} \hookrightarrow \dots$$

  of canonical embeddings such that the remainders $\mathbf{KL}_{p+1} \setminus f_p\left(\mathbf{KL}_p\right)$ are infinite discrete.
- $f_p\left(\mathbf{KL}_p\right)$ meets every nonempty closed set and every nonempty open set of $\mathbf{KL}_{p+1}$.
- For each positive integer p, $\mathbf{KL}_p$ is a connected Alexandroff space such that every continuous injection from the space into itself is a homeomorphism.
- We also provide details regarding the dimensions of $\mathbf{KL}_p$ and its degree of homogeneity.

2. Preliminaries

It can be readily verified that $\mathcal{K}_p$ defines an Alexandroff topology on the set of integers $\mathbb{Z}$. We denote the smallest $\mathcal{K}_p$-open set containing any integer $x \in \mathbb{Z}$ as $\mathcal{N}_p(x)$. Additionally, for any subset A of $\mathbb{Z}$, the $\mathcal{K}_p$-topological closure of A is represented by $\overline{A}^p$.

The following proposition is a straightforward implication of the definition provided in 1.1.

Proposition 2.1. Let p be a positive integer, x and i be integers such that $1 \leq i < p$. Then, the following properties hold.

(1) $x \equiv 0 \pmod{2p}$ if and only if $\overline{\{x\}}^p = \{x\}$ (equivalently, $\mathcal{N}_p(x) = [x - p, x + p]_{\mathbb{Z}}$).
(2) $x \equiv -i \pmod{2p}$ if and only if $\overline{\{x\}}^p = [x, x + i]_{\mathbb{Z}}$ (equivalently, $\mathcal{N}_p(x) = [x + i - p, x]_{\mathbb{Z}}$).
(3) $x \equiv i \pmod{2p}$ if and only if $\overline{\{x\}}^p = [x - i, x]_{\mathbb{Z}}$ (equivalently, $\mathcal{N}_p(x) = [x, x - i + p]_{\mathbb{Z}}$).
(4) $x \equiv p \pmod{2p}$ if and only if $\overline{\{x\}}^p = [x - p, x + p]_{\mathbb{Z}}$ (equivalently, $\mathcal{N}_p(x) = \{x\}$).

Remark 2.2. Let $\leq_p$ represent the ordering on $\mathbb{Z}$ that is induced by the $\mathcal{K}_p$-topology. In this context, the partially ordered set (poset) $(\mathbb{Z}, \leq_p)$ looks like:

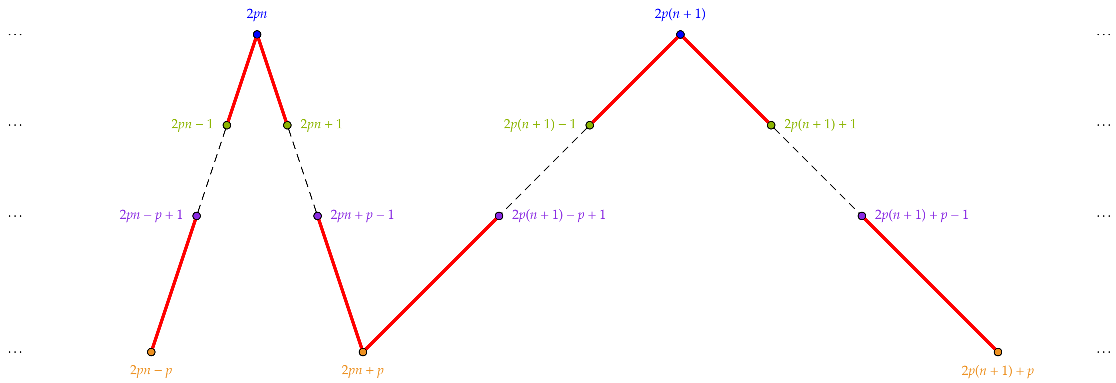

Figure 2: The poset $(\mathbb{Z}, \leq_p)$.

The enumeration of the cardinalities $|\overline{\{x\}}^p|$ and $|\mathcal{N}_p(x)|$ is determined by the congruence class of x modulo $2p$.

Proposition 2.3. *Consider the poset $(\mathbb{Z}, \leq_p)$. Then the following properties hold.*

(1) *Two integers a and b are comparable in $(\mathbb{Z}, \leq_p)$ if and only if they belong to a maximal chain of the form:*

$$x - p \leq_p x - p + 1 \leq_p \ldots \leq_p x - 1 \leq_p x$$

or

$$x \leq_p x + 1 \leq_p \ldots \leq_p x + p - 1 \leq_p x + p,$$

for some $x \equiv 0 \pmod{2p}$.

(2) *An integer $x \in \mathbb{Z}$ is maximal (resp. minimal) in $(\mathbb{Z}, \leq_p)$ if and only if $x \equiv 0 \pmod{2p}$ (resp. $x \equiv p \pmod{2p}$).*

(3) *$x \equiv 0 \pmod{2p}$ if and only if $|\overline{\{x\}}^p| = 1$ (equivalently, $|\mathcal{N}_p(x)| = 2p + 1$).*

(4) *$x \equiv p \pmod{2p}$ if and only if $|\overline{\{x\}}^p| = 2p + 1$ (equivalently, $|\mathcal{N}_p(x)| = 1$).*

(5) *For $x \equiv i \pmod{2p}$, with $i \in [1, p-1]_{\mathbb{Z}}$, it holds that $|\overline{\{x\}}^p| = i + 1$ and $|\mathcal{N}_p(x)| = p - i + 1$.*

(6) *For $x \equiv -i \pmod{2p}$, with $i \in [1, p-1]_{\mathbb{Z}}$, it holds that $|\overline{\{x\}}^p| = i + 1$ and $|\mathcal{N}_p(x)| = p - i + 1$.*

Before exploring continuous functions from $\mathbf{KL}_p$ to $\mathbf{KL}_q$, it is essential to revisit a fundamental concept in topology. Consider a function f from an Alexandroff space $(X_1, \mathcal{T}_1)$ to another Alexandroff space $(X_2, \mathcal{T}_2)$. In this setting, $\leq_i$ denotes the preorder induced by the topology $\mathcal{T}_i$ on the set X_i, for $i = 1, 2$. The function f is continuous if and only if it preserves the preorder, meaning $f : (X_1, \leq_1) \to (X_2, \leq_2)$ respects the ordering relations.

Thus, for a function $f : \mathbf{KL}_p \to \mathbf{KL}_q$ to be continuous, it is both necessary and sufficient that for all integers x and y, the implication

$$y \leq_p x \Longrightarrow f(y) \leq_q f(x)$$

holds. If $x \equiv p \pmod{2p}$, this implication holds trivially since, in this case, x is a minimal element of $(\mathbb{Z}, \leq_p)$, and therefore $y = x$. Consequently, our focus will be on cases where $x \not\equiv p \pmod{2p}$.

Proposition 2.4. *Let p and q be positive integers. Consider a function $f : \mathbf{KL}_p \to \mathbf{KL}_q$. This function is continuous if and only if for any $x \in \mathbb{Z}$ satisfying $x \not\equiv p \pmod{2p}$, the following conditions are satisfied:*

(1) *If $x \equiv 0 \pmod{2p}$ and $f(x) \equiv 0 \pmod{2q}$, then for each $t \in [-p, p]_{\mathbb{Z}}$, $f(x) - q \leq f(x + t) \leq f(x) + q$.*

(2) *If* $x \equiv 0 \pmod{2p}$ *and* $f(x) \equiv -j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each* $t \in [-p,p]_{\mathbb{Z}}$*,* $f(x) + j - q \leq f(x+t) \leq f(x)$.
(3) *If* $x \equiv 0 \pmod{2p}$ *and* $f(x) \equiv j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each* $t \in [-p,p]_{\mathbb{Z}}$*,* $f(x) \leq f(x+t) \leq f(x) - j + q$.
(4) *If* $x \equiv 0 \pmod{2p}$ *and* $f(x) \equiv q \pmod{2q}$*, then for each* $t \in [-p,p]_{\mathbb{Z}}$*,* $f(x+t) = f(x)$.
(5) *If* $x \equiv -i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv 0 \pmod{2q}$*, then for each* $t \in [i-p,0]_{\mathbb{Z}}$*,* $f(x) - q \leq f(x+t) \leq f(x) + q$.
(6) *If* $x \equiv -i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv -j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each integer* $i - p \leq t \leq 0$*,* $f(x) + j - q \leq f(x+t) \leq f(x)$.
(7) *If* $x \equiv -i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each* $t \in [i-p,0]_{\mathbb{Z}}$*,* $f(x) \leq f(x+t) \leq f(x) - j + q$.
(8) *If* $x \equiv -i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv q \pmod{2q}$*, then for each* $t \in [i-p,0]_{\mathbb{Z}}$*,* $f(x+t) = f(x)$.
(9) *If* $x \equiv i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv 0 \pmod{2q}$*, then for each* $t \in [0,p-i]_{\mathbb{Z}}$*,* $f(x) - q \leq f(x+t) \leq f(x) + q$.
(10) *If* $x \equiv i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv -j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each* $t \in [0,p-i]_{\mathbb{Z}}$*,* $f(x) + j - q \leq f(x+t) \leq f(x)$.
(11) *If* $x \equiv i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv j \pmod{2q}$ *for some* $j \in [1,q)_{\mathbb{Z}}$*, then for each* $t \in [0,p-i]_{\mathbb{Z}}$*,* $f(x) \leq f(x+t) \leq f(x) - j + q$.
(12) *If* $x \equiv i \pmod{2p}$ *for some* $i \in [1,p)_{\mathbb{Z}}$ *and* $f(x) \equiv q \pmod{2q}$*, then for each* $t \in [0,p-i]_{\mathbb{Z}}$*,* $f(x+t) = f(x)$.

When $p = q = 1$, the derived result corresponds to a specific case previously established. This is essentially a restatement of a Melin's result on continuity, referenced as [22, Lemma 2].

Proposition 2.5 ([22]). *A function* $f : \mathbf{KL} \to \mathbf{KL}$ *is continuous if and only if the following properties are satisfied for every even integer* x.

1. *If* $f(x)$ *is even, then* $f(\{x-1, x, x+1\}) \subseteq \{f(x)-1, f(x), f(x)+1\}$.
2. *If* $f(x)$ *is odd, then* $f(x \pm 1) = f(x)$.

Recall from Herman [13, Theorem 4.2.2.] that an Alexandroff space $(X, \mathcal{T})$ is connected if and only if, for any two distinct elements x and y in X, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ of elements in X such that each pair of consecutive elements is $\mathcal{T}$-adjacent. This means for every i in the set $\{0, \ldots, n-1\}$, either x_i is in the closure of $\{x_{i+1}\}$ or x_{i+1} is in the closure of $\{x_i\}$.

Moreover, it is clear that any Alexandroff space $(X, \mathcal{T})$ is locally connected. This is because the smallest neighborhood $\mathcal{N}(x)$ of any point x is connected. For any two points a, b in $\mathcal{N}(x)$, the sequence (a, x, b) forms a chain of $\mathcal{T}$-adjacent elements within $\mathcal{N}(x)$.

Considering that every neighborhood $\mathcal{N}_p(x)$ forms a connected subspace in $\mathbf{KL}_p$, and given that $\mathbb{Z}$ is the union of these neighborhoods, expressed as $\mathbb{Z} = \bigcup \left[\mathcal{N}_p(2np) \colon n \in \mathbb{Z}\right]$, and as $\mathcal{N}_p(2np) \cap \mathcal{N}_p(2(n+1)p) = \{2np + p\}$, we can deduce the following result.

Proposition 2.6. For each positive integer p, the space $\mathbf{KL}_p$ is connected.

We conclude this preliminary section by introducing a canonical dense embedding of $\mathbf{KL}_p$ into $\mathbf{KL}_q$ for $p \leq q$, denoted as $\mathbf{KL}_p \hookrightarrow \mathbf{KL}_q$. The embedding is facilitated by the function $\varphi_{p,q} : \mathbb{Z} \to \mathbb{Z}$, defined as follows:

$$\varphi_{p,q}(x) = \begin{cases} q\left(\frac{x+p}{p}\right) - q, & \text{if } x \equiv -p \pmod{2p}, \\ \\ q\left(\frac{x-i}{p}\right) + i, & \text{if } x \equiv i \pmod{2p}, \text{ for } -p+1 \leq i \leq p-1. \end{cases}$$

In the special case where $q = p + 1$, this function is denoted by φ_p.

To provide a local visualization of this mapping, consider the following diagram.

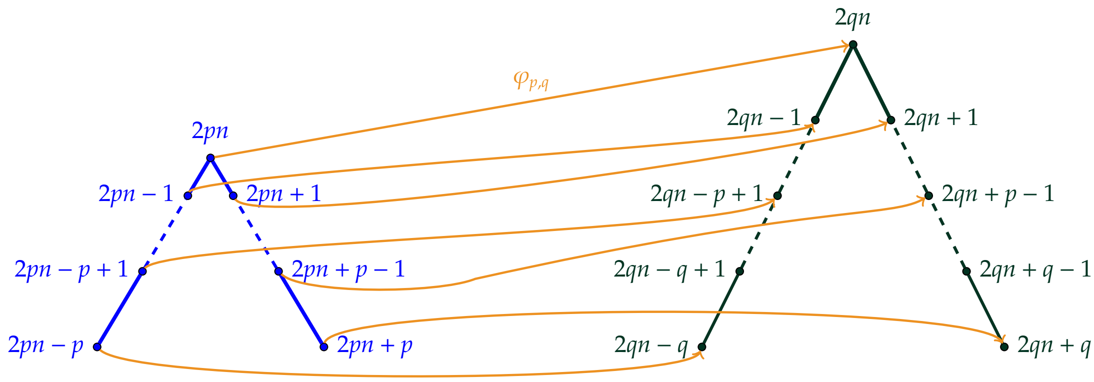

Figure 3: The assignment of $\varphi_{p,q}$.

According to [9], a subset S of a space X is defined as *strongly dense* (or *très dense* in French) if it intersects every nonempty locally closed subset (i.e. intersection of an open set and a closed set) within X.

Definition 2.7. *A subset D of a space X is termed sufficiently dense if it intersects with every nonempty open set as well as every nonempty closed set of X.*

Remark 2.8. *It is clear that the following implications hold:*

$$S \text{ is strongly dense in } X \implies S \text{ is sufficiently dense in } X \implies S \text{ is dense in } X,$$

and none of these implications is reversible.

For instance, the set of all rational numbers $\mathbb{Q}$ is dense in the real line $\mathbb{R}$ (when equipped with the standard topology) but is not strongly dense. To show the non-reversibility of the first implication, consider a partially ordered set $(X, \leq)$ that includes at least one minimal and one maximal element, and contains three comparable elements $x \leq y \leq z$. Equip X with the Alexandroff topology induced by the ordering $\leq$. Define S as $\mathrm{Min}(X) \cup \mathrm{Max}(X)$, the set comprising all minimal and maximal elements of X. While S is sufficiently dense in X, it is not strongly dense since S does not intersect the locally closed set $\{y\}$ in X.

Proposition 2.9. *Let p and q be positive integers with $p \leq q$. Then, the following properties hold.*

(1) *The function $\varphi_{p,q}$ can be expressed as the composition of functions: $\varphi_{p,q} = \varphi_{q-1} \circ \varphi_{q-2} \circ \ldots \circ \varphi_p$.*
(2) *The function $\varphi_{p,q}$ is a topological embedding, and the image $\varphi_{p,q}(\mathbb{Z})$ is sufficiently dense in $\mathbf{KL}_q$.*
(3) *The remainder $R_p := \mathbb{Z} \setminus \varphi_p(\mathbb{Z})$ forms an infinite discrete subspace of $\mathbf{KL}_{p+1}$.*

Proof.

(1) This part is straightforward.
(2) The function $\varphi_{p,q}$ preserves order from $(\mathbb{Z}, \leq_p)$ to $(\mathbb{Z}, \leq_q)$, making it a continuous injection from $\mathbf{KL}_p$ into $\mathbf{KL}_q$. Additionally, for every $x \in \mathbb{Z}$, we have

$$\varphi_{p,q}\left(\mathcal{N}_p(x)\right) = \mathcal{N}_q(\varphi_{p,q}(x)) \cap \varphi_{p,q}(\mathbb{Z}).$$

Hence, $\varphi_{p,q}$ is a topological embedding.
Furthermore, every $y \in \mathbb{Z}$ in $\mathcal{N}_q(y)$ includes a minimal element of the poset $(\mathbb{Z}, \leq_q)$ (specifically, an integer congruent to q modulo $2q$). Such minimal elements belong to $\varphi_{p,q}(\mathbb{Z})$. Similarly, $\overline{\{y\}}^q$ contains a maximal element of $(\mathbb{Z}, \leq_q)$ (an integer congruent to 0 modulo $2q$). Consequently, $\varphi_{p,q}(\mathbb{Z})$ is sufficiently dense in $\mathbf{KL}_q$.

(3) Define

$$R_p: \;=\; \mathbb{Z} \setminus \varphi_p(\mathbb{Z}) \;=\; \{x \in \mathbb{Z} : x \equiv p \pmod{2(p+1)} \text{ or } x \equiv -p \pmod{2(p+1)}\}.$$

For any $x \in R_p$, we find that

$$\mathcal{N}_{p+1}(x) = \begin{cases} \{x, x-1\}, & \text{if } x \equiv -p \pmod{2(p+1)}, \\ \{x, x+1\}, & \text{if } x \equiv p \pmod{2(p+1)}. \end{cases}$$

This implies that $\mathcal{N}_{p+1}(x) \cap R_p = \{x\}$. Therefore, R_p is a discrete subspace of $\mathbf{KL}_{p+1}$.

□

3. Continuous Injections of $\mathbf{KL}_p$ into Itself

This section focuses on continuous, one-to-one functions from $\mathbf{KL}_p$ to itself.

Lemma 3.1. *Consider a continuous injection $f : \mathbf{KL}_p \to \mathbf{KL}_p$ and let x be an integer. Then the following properties hold.*

(1) $x \equiv 0 \pmod{2p}$ *if and only if* $f(x) \equiv 0 \pmod{2p}$.
(2) $x \equiv p \pmod{2p}$ *if and only if* $f(x) \equiv p \pmod{2p}$.
(3) *If* $x \equiv 0 \pmod{2p}$, *then the set* $\{f(x-p), f(x+p)\}$ *equals* $\{f(x)-p, f(x)+p\}$.
(4) *If* $x \equiv 0 \pmod{2p}$ *and* $f(x-p) = f(x)-p$, *then for all i in* $[0,p]$, *it holds that* $f(x-i) = f(x)-i$ *and* $f(x+i) = f(x)+i$.
(5) *If* $x \equiv 0 \pmod{2p}$ *and* $f(x-p) = f(x)+p$, *then for all i in* $[0,p]$, *it holds that* $f(x-i) = f(x)+i$ *and* $f(x+i) = f(x)-i$.

Proof.

(1) Assume that $x \equiv 0 \pmod{2p}$. By Proposition 2.3, we have $|\mathcal{N}_p(x)| = 2p+1$. By continuity, it follows that $f(\mathcal{N}_p(x)) \subseteq \mathcal{N}_p(f(x))$. Since f is injective, we have $|f(\mathcal{N}_p(x))| = 2p+1$. Therefore, $|\mathcal{N}_p(f(x))| \geq 2p+1$, and applying Proposition 2.3 once more, we conclude that $|\mathcal{N}_p(f(x))| = 2p+1$. Consequently, $f(x) \equiv 0 \pmod{2p}$.

Conversely, if $f(x) \equiv 0 \pmod{2p}$, Proposition 2.3 implies that $|\overline{\{f(x)\}}^p| = 1$. Continuity ensures that $f(\overline{\{x\}}^p) \subseteq \overline{\{f(x)\}}^p = \{f(x)\}$. Furthermore, since f is injective, it follows that $|\overline{\{x\}}^p| = 1$ and, consequently, $x \equiv 0 \pmod{2p}$.

(2) By Proposition 2.3, we have $x \equiv p \pmod{2p}$ if and only if $|\overline{\{x\}}| = 2p+1$. Assume $x \equiv p \pmod{2p}$. Since $f(\overline{\{x\}}^p) \subseteq \overline{\{f(x)\}}^p$ and f is injective, it follows that $|\overline{\{f(x)\}}^p| = 2p+1$. Therefore, $f(x) \equiv p \pmod{2p}$.

Conversely, suppose $f(x) \equiv p \pmod{2p}$. By Proposition 2.3, we have $|\mathcal{N}_p(f(x))| = 1$. Since $f(\mathcal{N}_p(x)) \subseteq \mathcal{N}_p(f(x))$ and f is injective, we deduce that $|\mathcal{N}_p(x)| = 1$. Consequently, $x \equiv p \pmod{2p}$.

(3) Assume that $x \equiv 0 \pmod{2p}$. By (1) and Proposition 2.3, we have $|\mathcal{N}_p(f(x))| = 2p+1 = |\mathcal{N}_p(x)|$. Since $f(\mathcal{N}_p(x)) \subseteq \mathcal{N}_p(f(x))$ and f is injective, it follows that $f(\mathcal{N}_p(x)) = \mathcal{N}_p(f(x))$. Consequently, $f(x-p), f(x+p) \in [f(x)-p, f(x)+p]_{\mathbb{Z}}$. Given that $x-p \equiv p \pmod{2p}$ and $x+p \equiv p \pmod{2p}$, it follows from (2) that $f(x-p) \equiv p \pmod{2p}$ and $f(x+p) \equiv p \pmod{2p}$. Thus, $\{f(x-p), f(x+p)\} = \{f(x)-p, f(x)+p\}$.

In the remainder of the proof, if we assume $x \equiv 0 \pmod{2p}$, then by (3) we have $\{f(x-p), f(x+p)\} = \{f(x)-p, f(x)+p\}$. This leads to two cases to consider: either $f(x-p) = f(x)-p$ or $f(x-p) = f(x)+p$. These cases will be discussed below in (4) and (5).

(4) Suppose that $x \equiv 0 \pmod{2p}$ and $f(x-p) = f(x)-p$. Then, by (3), since $\{f(x-p), f(x+p)\} = \{f(x)-p, f(x)+p\}$ and f is one-to-one, we deduce that $f(x+p) = f(x)+p$. By continuity, we have $f(\mathcal{N}_p(x)) \subseteq \mathcal{N}_p(f(x))$, and since the two sets have the same cardinality $2p+1$, it follows that $f(\mathcal{N}_p(x)) = \mathcal{N}_p(f(x))$.

As f is order-preserving, we obtain two chains:

$$C : f(x)-p = f(x-p) \leq_p f(x-p+1) \leq_p \ldots \leq_p f(x-1) \leq_p f(x) \quad \text{and}$$

$$C' : f(x) + p = f(x+p) \leq_p f(x+p-1) \leq_p \ldots \leq_p f(x+1) \leq_p f(x)$$

of length $p+1$ in $(\mathcal{N}_p(f(x)), \leq_p)$. This poset contains exactly two maximal chains, namely:

$$C_1 : f(x) - p \leq_p f(x) - p + 1 \leq_p \ldots \leq_p f(x) - 1 \leq_p f(x) \quad \text{and}$$

$$C_2 : f(x) + p \leq_p f(x) + p - 1 \leq_p \ldots \leq_p f(x) + 1 \leq_p f(x),$$

implying that C and C' match C_1 and C_2, respectively. Therefore,

$$f(x-i) = f(x) - i \quad \text{and} \quad f(x+i) = f(x) + i \quad \text{for all } i \in [0,p]_{\mathbb{Z}}.$$

(5) Although the proof is similar to that of (4), we will include it here for completeness.

Suppose that $x \equiv 0 \pmod{2p}$ and $f(x-p) = f(x)+p$. Then, by (3), since $\{f(x-p), f(x+p)\} = \{f(x)-p, f(x)+p\}$ and f is one-to-one, we deduce that $f(x+p) = f(x) - p$. By continuity, we have $f(\mathcal{N}_p(x)) \subseteq \mathcal{N}_p(f(x))$, and since the two sets have the same cardinality $2p+1$, it follows that $f(\mathcal{N}_p(x)) = \mathcal{N}_p(f(x))$.

As f is order-preserving, we obtain two chains:

$$\mathcal{D} : f(x) - p = f(x+p) \leq_p f(x+p-1) \leq_p \ldots \leq_p f(x+1) \leq_p f(x)$$

and

$$\mathcal{D}' : f(x) + p = f(x-p) \leq_p f(x-p+1) \leq_p \ldots \leq_p f(x-1) \leq_p f(x)$$

of length $p+1$ in $(\mathcal{N}_p(f(x)), \leq_p)$. This poset contains exactly two maximal chains, namely:

$$\mathcal{D}_1 : f(x) - p \leq_p f(x) - p + 1 \leq_p \ldots \leq_p f(x) - 1 \leq_p f(x) \quad \text{and}$$

$$\mathcal{D}_2 : f(x) + p \leq_p f(x) + p - 1 \leq_p \ldots \leq_p f(x) + 1 \leq_p f(x),$$

implying that $\mathcal{D}$ and $\mathcal{D}'$ match $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Therefore,

$$f(x+i) = f(x) - i \quad \text{and} \quad f(x-i) = f(x) + i \quad \text{for all } i \in [0,p]_{\mathbb{Z}}.$$

□

Next, we will demonstrate that every p-Khalimsky topology possesses a "kind of reversibility property."

According to [24], a topological space $(X, \mathcal{T})$ is said to be *reversible* if every continuous bijection from X onto itself is a homeomorphism.

In 1976, the authors of [5] provided necessary and sufficient conditions under which continuous bijections of a manifold onto itself are homeomorphisms. Later, in 1983, the same authors introduced and studied the concept of "strongly reversible manifolds" in [14]. Specifically, an n-manifold M is defined as strongly reversible if every continuous bijection from M into an n-manifold is a homeomorphism. Consequently, every compact manifold is strongly reversible. It was also shown that every strong reversible n-manifold is reversible, but the converse does not hold (see [14]).

If M is strongly reversible and $f : M \longrightarrow M$ is a continuous injection, f may not necessarily be a homeomorphism (see the next remark). This observation motivates the following concept.

Definition 3.2. *A plus-reversible space is defined as a space where every continuous injection into itself is a homeomorphism.*

Remark 3.3. *While every plus-reversible space is reversible, the converse does not necessarily hold. For instance, any compact Hausdorff space is reversible [24], but may not be plus-reversible. Consider the interval $I = [a,b]$ on the real line (with $a < b$), equipped with the standard topology. Let $J = [c,d] \subset I$ be a sub-interval of I. The function $\theta : t \mapsto \frac{d-c}{b-a}t + \frac{bc-ad}{b-a}$ is a continuous injection of I into itself. However, $\theta(I) = J \neq I$, meaning that θ is not surjective, and thus the space is not plus-reversible.*

In terms of strong reversibility, it is evident that $M = [a,b]$ is a strongly reversible 1-manifold that is not plus-reversible.

Question 3.4. *Explore the conditions under which a reversible space is also plus-reversible.*

In this paper, we adopt the perspective of Melin (see [23]) about locally finite spaces. A space $(X, \mathcal{T})$ is referred to as *locally finite* if every point $x \in X$ has a finite neighborhood and the topological closure $\overline{\{x\}}^{\mathcal{T}}$ is also finite. It is worth noting that if if every point $x \in X$ has a finite neighborhood, then X is an Alexandroff space.

Lemma 3.5. *Consider a connected, locally finite topological space $(X, \mathcal{T})$, and let $\theta : X \to X$ be a continuous injection. Then θ is a homeomorphism if and only if, for every x in X,*

$$\left|\overline{\{x\}}^{\mathcal{T}}\right| = \left|\overline{\{\theta(x)\}}^{\mathcal{T}}\right| \textit{ and } \left|\mathcal{N}_{\mathcal{T}}(x)\right| = \left|\mathcal{N}_{\mathcal{T}}(\theta(x))\right|.$$

Proof. Suppose that for every x in X, it holds that

$$\left|\overline{\{x\}}\right| = \left|\overline{\{\theta(x)\}}\right| \text{ and } \left|\mathcal{N}(x)\right| = \left|\mathcal{N}(\theta(x))\right|.$$

We aim to show that θ is a clopen map. Given that θ is continuous, it follows that

$$\theta(\overline{\{x\}}) \subseteq \overline{\{\theta(x)\}} \text{ and } \theta(\mathcal{N}(x)) \subseteq \mathcal{N}(\theta(x)).$$

Since X is locally finite and θ is injective, our assumption leads to the equalities:

$$\theta(\overline{\{x\}}) = \overline{\{\theta(x)\}} \text{ and } \theta(\mathcal{N}(x)) = \mathcal{N}(\theta(x)),$$

for every x in X. Consequently, as both the source and target spaces are Alexandroff spaces, θ is a clopen map. Therefore, $\theta(X)$ is clopen in X. Given that X is also connected, this implies $\theta(X) = X$. Hence, θ is a clopen continuous bijection, and thus a homeomorphism.

Conversely, if we assume that θ is a homeomorphism, then for every $x \in X$, it follows that $\overline{\{x\}}^{\mathcal{T}} = \overline{\{\theta(x)\}}^{\mathcal{T}}$ and $\mathcal{N}_{\mathcal{T}}(x) = \mathcal{N}_{\mathcal{T}}(\theta(x))$. By considering the cardinalities, we obtain $\left|\overline{\{x\}}^{\mathcal{T}}\right| = \left|\overline{\{\theta(x)\}}^{\mathcal{T}}\right|$ and $\left|\mathcal{N}_{\mathcal{T}}(x)\right| = \left|\mathcal{N}_{\mathcal{T}}(\theta(x))\right|$. □

Theorem 3.6. *For every positive integer p, $\mathbf{KL}_p$ is a plus-reversible space.*

Proof. As per Lemma 3.5, to establish this, it is sufficient to show that for a continuous injective map $\theta : \mathbf{KL}_p \to \mathbf{KL}_p$, the following holds for every x in X:

$$\left|\overline{\{x\}}\right| = \left|\overline{\{\theta(x)\}}\right| \text{ and } \left|\mathcal{N}(x)\right| = \left|\mathcal{N}(\theta(x))\right|.$$

We consider three cases:

- Case 1: If $x \equiv 0 \pmod{2p}$, Lemma 3.1 suggests $\theta(x) \equiv 0 \pmod{2p}$. Using Proposition 2.3, it follows that

$$\left|\overline{\{x\}}^{p}\right| = 1 = \left|\overline{\{\theta(x)\}}^{p}\right| \text{ and } \left|\mathcal{N}_p(x)\right| = 2p + 1 = \left|\mathcal{N}_p(\theta(x))\right|.$$

- Case 2: For $x \equiv p \pmod{2p}$, Lemma 3.1 implies $\theta(x) \equiv p \pmod{2p}$. Hence, by Proposition 2.3, we have

$$\left|\overline{\{x\}}^{p}\right| = 2p + 1 = \left|\overline{\{\theta(x)\}}^{p}\right| \text{ and } \left|\mathcal{N}_p(x)\right| = 1 = \left|\mathcal{N}_p(\theta(x))\right|.$$

- Case 3: Suppose $x \equiv \varepsilon i \pmod{2p}$ for some i in $[1, p-1]_{\mathbb{Z}}$ and $\varepsilon \in \{-1, 1\}$. According to Lemma 3.1, $\theta(x) \equiv \varepsilon i \pmod{2p}$. Again, from Proposition 2.3, we deduce

$$\left|\overline{\{x\}}^{p}\right| = i + 1 = \left|\overline{\{\theta(x)\}}^{p}\right| \text{ and } \left|\mathcal{N}_p(x)\right| = p - i + 1 = \left|\mathcal{N}_p(\theta(x))\right|.$$

□

4. The Group $\mathcal{H}(\mathbf{KL}_p)$

Recall from [25] that the infinite dihedral group $\mathcal{D}_\infty$ is defined by the presentation

$$\langle s, t \mid t^2 = 1, tst^{-1} = s^{-1} \rangle,$$

where s has an infinite order. This means that $\mathcal{D}_\infty$ is generated by two elements s and t, such that t is of order 2, s has infinite order, and they satisfy the relation $tst^{-1} = s^{-1}$. It is also well-known that any infinite group generated by two elements of order 2 is isomorphic to $\mathcal{D}_\infty$; see the expository paper by Keith Conrad [4] for further details.

A concrete realization of $\mathcal{D}_\infty$ is provided by the affine group $\mathbf{Aff}(\mathbb{Z})$, consisting of all affine functions $f : \mathbb{Z} \to \mathbb{Z}$ of the form $f(x) = ax + b$, where $a = \pm 1$ and $b \in \mathbb{Z}$, with the group operation being composition.

In this section, we aim to establish that the group of homeomorphisms $\mathcal{H}(\mathbf{KL}_p)$ is isomorphic to $\mathcal{D}_\infty$.

Remark 4.1. *Consider the reflection $\sigma : \mathbf{KL}_p \to \mathbf{KL}_p$ defined by $\sigma(x) = -x$. For any integer a such that $a \equiv 0 \pmod{2p}$, we define the translation $\tau_a : \mathbf{KL}_p \to \mathbf{KL}_p$ by $\tau_a(x) = x + a$. It is clear that both τ_a and σ are continuous bijections, and thus belong to the group of homeomorphisms $\mathcal{H}(\mathbf{KL}_p)$ by Theorem 3.6. Moreover, the following properties hold:*

1. *$\sigma^2 = \mathbf{1}$ and $\sigma\tau_a\sigma^{-1} = \tau_{-a} = (\tau_a)^{-1}$ (equivalently, $(\sigma\tau_a)^2 = \mathbf{1}$).*
2. *The order of τ_a is infinite for every $a \neq 0$ such that $a \equiv 0 \pmod{2p}$.*

Lemma 4.2. *Consider a homeomorphism $f : \mathbf{KL}_p \to \mathbf{KL}_p$. Then the following properties hold.*

(1) *If $f(0) = 0$ and $f(p) = p$, then f is the identity mapping, denoted by $\mathbf{1}$.*
(2) *If $f(0) = 0$ and $f(p) = -p$, then f is the reflection σ, as defined in Remark 4.1.*

Proof.

(1) We will prove by induction on the nonnegative integer n that for every homeomorphism f such that $f(0) = 0$ and $f(p) = p$, it holds that $f(x) = x$ for all $|x| \leq 2np + p$.
- For $n = 0$, Since $f(0) = 0$, by Lemma 3.1(3), we have

$$f(\{0 - p, 0 + p\}) = \{f(0) - p, f(0) + p\} = \{-p, p\}.$$

Given that $f(p) = p$, it follows that $f(-p) = -p$. Consequently, by Lemma 3.1(4), we obtain $f(0 - i) = f(0) - i$ and $f(0 + i) = f(0) + i$. Therefore, $f(x) = x$ for all $|x| \leq p$.
- Assume that for every $f \in \mathcal{H}(\mathbf{KL}_p)$ with $f(0) = 0$ and $f(p) = p$, we have $f(x) = x$ for all $|x| \leq 2np + p$. We will show that $f(t) = t$ for all $|t| \leq 2(n + 1)p + p$. It remains to prove that

$$f(t) = t \text{ for all } t \in [2(n + 1)p - p, 2(n + 1)p + p]_{\mathbb{Z}} \cup [-2(n + 1)p - p, -2(n + 1)p + p]_{\mathbb{Z}}.$$

Since f is a homeomorphism, we have

$$f(\overline{\{2np + p\}}^p) = \overline{\{f(2np + p)\}}^p = \overline{\{2np + p\}}^p = [2np, 2(n + 1)p]_{\mathbb{Z}}.$$

Therefore, $f([2np, 2(n + 1)p]_{\mathbb{Z}}) = [2np, 2(n + 1)p]_{\mathbb{Z}}$.
By comparing the maximal chains of the posets $\left([2np, 2(n + 1)p]_{\mathbb{Z}}, \leq_p\right)$ and $\left(f\left([2np, 2(n + 1)p]_{\mathbb{Z}}\right), \leq_p\right)$, we deduce that

$$2np = f(2np) \leq_p \ldots \leq_p f(2np + p) = 2np + p \text{ (by the induction hypothesis)}$$

coincides with

$$2np \leq_p \ldots \leq_p 2np + p,$$

and similarly,

$$2np + p = f(2np + p) \leq_p f(2np + p + 1) \leq_p \ldots \leq_p f(2(n + 1)p)$$

coincides with

$$2np + p \leq_p 2np + p + 1 \leq_p \ldots \leq_p 2(n + 1)p.$$

In particular, $f(2(n + 1)p) = 2(n + 1)p$. Taking $x = 2(n + 1)p$, we have $f(x) = x$ and $f(x - p) = f(x) - p$. Thus, by applying Lemma 3.1(4), we deduce that $f(x - i) = f(x) - i = x - i$ and $f(x + i) = f(x) + i = x + i$ for every $i \in [0, p]_{\mathbb{Z}}$. This implies that $f(t) = t$ for every $t \in [2(n + 1)p - p, 2(n + 1)p + p]_{\mathbb{Z}}$.
Now, we will show that $f(t) = t$ for every $t \in [-2(n+1)p - p, -2np - p]_{\mathbb{Z}}$. Since f is a homeomorphism, we have

$$f(\overline{\{-2np - p\}}^p) = \overline{\{f(-2np - p)\}}^p = \overline{\{-2np - p\}}^p = [-2(n + 1)p, -2np]_{\mathbb{Z}}.$$

Thus, $f([-2(n + 1)p, -2np]_{\mathbb{Z}}) = [-2(n + 1)p, -2np]_{\mathbb{Z}}$.
By the induction hypothesis,

$$-2np - p = f(-2np - p) \leq_p -2np - p + 1 = f(-2np - p + 1) \leq_p \ldots \leq_p f(-2np + p) = -2np$$

is a maximal chain of $f([-2(n + 1)p, -2np]_{\mathbb{Z}})$ coinciding with the maximal chain

$$-2np - p \leq_p -2np - p + 1 \leq_p \ldots \leq_p -2np$$

of $[-2(n + 1)p, -2np]_{\mathbb{Z}}$. Similarly, the second maximal chain,

$$-2np - p = f(-2np - p) \leq_p f(-2np - p - 1) \leq_p \ldots \leq_p f(-2(n + 1)p),$$

matches the maximal chain

$$-2np - p \leq_p -2np - p - 1 \ldots \leq_p -2(n + 1)p$$

of the poset $[-2(n + 1)p, -2np]_{\mathbb{Z}}$. In particular, we have $f(-2(n + 1)p) = -2(n + 1)p$.
Now, taking $x = -2(n + 1)p$, we obtain $f(x) = x$ and $f(x + p) = f(-2np - p) = -2np - p = f(x) + p$. By Lemma 3.1(3), we conclude that $f(x - p) = f(x) - p$. Furthermore, according to Lemma 3.1(4), for every $i \in [0, p]$, we deduce that $f(x - i) = f(x) - i = x - i$ and $f(x + i) = f(x) + i = x + i$. This shows that $f(t) = t$ for all $t \in [-2(n + 1)p - p, -2np - p]_{\mathbb{Z}}$, thus completing the induction.

(2) Let $g = \sigma f$. Then g is a homeomorphism satisfying $g(0) = 0$ and $g(p) = -f(p) = p$. Therefore, by the previous step (1), $g = \mathbf{1}$, and consequently, $f = \sigma$.

□

Theorem 4.3. The group of autohomeomorphisms $\mathcal{H}(\mathbf{KL}_p)$ is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$.

Proof. We let σ and $\tau := \tau_{2p}$ be the elements of $\mathcal{H}(\mathbf{KL}_p)$, as defined in Remark 4.1.
We will show that σ and τ generate $\mathcal{H}(\mathbf{KL}_p)$. More precisely, we establish the equality:

$$\mathcal{H}(\mathbf{KL}_p) = \{\tau^k : k \in \mathbb{Z}\} \cup \{\tau^k \sigma : k \in \mathbb{Z}\}.$$

Consider $f \in \mathcal{H}(\mathbf{KL}_p)$, and let $a := f(0)$. By Lemma 3.1, $a \equiv 0 \pmod{2p}$, which means $a = 2kp$ for some integer k. Defining $g = \tau^{-k} f = \tau_{-a} f$, we have $g \in \mathcal{H}(\mathbf{KL}_p)$ and $g(0) = 0$. Lemma 3.1 (3) implies $\{g(-p), g(p)\} = \{-p, p\}$. We consider two cases:
Case 1: If $g(p) = p$, then Lemma 4.2 leads to $g = \mathbf{1}$. Consequently, $f = \tau_a = \tau^k$.
Case 2: If $g(p) = -p$, Lemma 4.2 implies $g = \sigma$. Therefore, $f = \tau_a \sigma = \tau^k \sigma$.

Thus, σ and τ indeed generate $\mathcal{H}(\mathbf{KL}_p)$, write $\mathcal{H}(\mathbf{KL}_p) = \langle \sigma, \tau \rangle$. But as $\tau = \sigma(\sigma\tau)$, we deduce that σ and $\sigma\tau$ generate $\mathcal{H}(\mathbf{KL}_p)$. From Remark 4.1, we have

$$\sigma^2 = \mathbf{1} = (\sigma\tau)^2.$$

Consequently, $\mathcal{H}(\mathbf{KL}_p)$ is an infinite group generated by two elements (σ and $\sigma\tau$) of order 2. Therefore $\mathcal{H}(\mathbf{KL}_p)$ is isomorphic to $\mathcal{D}_\infty$.
□

We conclude this section with the following question.

Question 4.4 (The Digital plane). *Characterize the group of autohomeomorphisms of the product space* $\mathbf{KL}_p \times \mathbf{KL}_p$.

5. Degree of Homogeneity and Dimensions of $\mathbf{KL}_p$

(i) *The Hight of a Poset*

Let $(P, \leq)$ be a poset, and let C be a finite chain in $(P, \leq)$ (i.e., a totally ordered subset of P). The *length* of C is defined as $|C| - 1$. The *height* of $(P, \leq)$, denoted by $\mathrm{ht}(P)$, is the supremum in $\{-1, 0, 1, 2, \ldots, +\infty\}$ of the lengths of its chains. For $x \in P$, the *height of* x, denoted by $\mathrm{ht}(x)$, is defined as the supremum of the lengths of all chains whose greatest element is x. Clearly, $\mathrm{ht}(P) = \sup(\{\mathrm{ht}(x) : x \in P\})$.

When $P = \emptyset$, the only ordering on P is the empty ordering, and the only chain of $(P, \leq)$ is $C = \emptyset$, its length is $|C| - 1 = -1$. Hence $\mathrm{ht}(P) = -1$.

The T_0-*identification* (or *Kolmogorov quotient*) of a topological space refers to a process that transforms a given space into a T_0-*space*. Given a topological space X, the T_0-identification is constructed by defining an equivalence relation $\sim$ on X, where two points $x, y \in X$ are considered equivalent (i.e., $x \sim y$) if and only if they are *topologically indistinguishable* (this means that the closure of $\{x\}$ is the same as the closure of $\{y\}$).

The height of a non-necessarily T_0-space X is defined to be that of its T_0-identification, denoted by $T_0(X)$; that is, we define: $\mathrm{ht}(X) = \mathrm{ht}(T_0(X))$.

(ii) *Degree of Homogeneity of a Space*

The *degree of homogeneity* of a topological space measures the extent to which the space is uniform or symmetric with respect to its points.

A topological space X is said to be *homogeneous* if, for any two points $x, y \in X$, there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$. This means that the space exhibits the same structure from the perspective of any point. The *degree of homogeneity* of X, denoted by $d_H(X)$, is defined as the cardinality of the quotient set $X/\mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ denotes the group of homeomorphisms of X. This quotient set represents the collection of orbits of points under the action of the homeomorphism group. This concept is useful in understanding the symmetry properties of topological spaces and plays a role in various areas of topology and geometry.

The following result links the height of a space X with the degree of homogeneity $d_H(X)$.

Theorem 5.1. *If* $(X, \mathcal{T})$ *is a* T_0*-space, with finite height, then*

$$d_H(X) \geq \mathrm{ht}(X) + 1.$$

Proof. Let $h = \mathrm{ht}(X)$ and $x_0 < x_1 < \ldots < x_h$ be a chain of elements of $(X, \leq)$ of length h. Then $\mathrm{ht}(x_i) = i$, for each i. For every $i \neq j$ in $\{0, 1, \ldots, h\}$, the orbits $\overline{x_i}$ and $\overline{x_j}$ under the action of the group $\mathcal{H}(X)$ are distinct, otherwise there exists $g \in \mathcal{H}(X)$, such that $x_i = g(x_j)$.

We claim that if $g \in \mathcal{H}(X)$, then $g : (X, \leq) \longrightarrow (X, \leq)$ is order-preserving. Indeed, assuming $a \leq b$ in X, we have $b \in \overline{\{a\}}$. Now as g is a homeomorphism $g(b) \in g\left(\overline{\{a\}}\right) = \overline{\{g(a)\}}$. So $g(a) \leq g(b)$. It follows that $g : (X, \leq) \longrightarrow (X, \leq)$ is an isomorphism of posets. As a result, the equality $x_i = g(x_j)$ implies that x_i and x_j have the same height, a contradiction. This shows that $d_H(X) \geq h + 1$. □

Example 5.2. *If X is a T_0-Alexandroff space with infinite height, the inequality $d_H(X) \geq \mathrm{ht}(X)+1$ does not necessarily hold.*

Indeed, consider the set of integers $X = \mathbb{Z}$ endowed with the usual ordering $\leq$. Then $\mathrm{ht}(X) = +\infty$. However, if $\mathcal{T}$ is the Alexandroff topology on X associated with $\leq$ (where the open sets are exactly $\emptyset$, X, and $(-\infty, n]_{\mathbb{Z}}$ for any integer n), then clearly $(X, \mathcal{T})$ is homogeneous. Since for any $n \leq m$ in X, we have $m = g(n)$, where $g : X \longrightarrow X$ is the homeomorphism given by $g(x) = x + (m - n)$, it follows that $d_H(X) = 1$.

Theorem 5.1 motivates the introduction of the following concept.

Definition 5.3. *A T_0-space X is termed well-leveled if it has a finite height and $d_H(X) = \mathrm{ht}(X) + 1$.*

The following proposition follows immediately from the above definition.

Proposition 5.4. *Let X be a T_0-space with finite height h. Then the following statements are equivalent.*

(*i*) *X is well-leveled.*

(*ii*) *The orbits of the action of $\mathcal{H}(X)$ on X are exactly the levels $L_0, L_1, \ldots, L_h$, where $h = \mathrm{ht}(X)$ and*

$$L_i = \{x \in X : ht(x) = i\}.$$

(*iii*) *$\mathcal{H}(X)$ acts transitively on every level L_i.*

Example 5.5. *For every positive integer n, the Euclidean space $\mathbb{R}^n$, endowed with the usual topology $\mathfrak{U}_n$, is well-leveled, as $d_H(\mathbb{R}^n) = 1$ and $\mathrm{ht}(\mathbb{R}^n) = 0$. More generally, every homogeneous T_1-space is well-leveled.*

Now, we present an additional property of the digital topology $\mathbf{KL}_p$.

Proposition 5.6. *For every positive integer p, the space $(\mathbb{Z}, \mathbf{KL}_p)$ is well-leveled.*

Proof. According to Remark 2.2, the poset $(\mathbb{Z}, \leq_p)$ has height p, with its levels given by:

$$L_i = \{x \in \mathbb{Z} : x \equiv (p - i) \pmod{2p}\}, \quad \text{for } i \in \{0, 1, \ldots, p\}$$

These levels form a partition of $\mathbb{Z}$. Moreover, if $f \in \mathcal{H}(\mathbf{KL}_p)$, then by Lemma 3.1, we have $f(L_i) \subseteq L_i$. Now, for any $x = 2pn + p - i$ and $y = 2pm + p - i$ in L_i, consider the function $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $g(z) = z + 2p(m - n)$; then clearly $g(x) = y$, and $g \in \mathcal{H}(\mathbf{KL}_p)$. Therefore, $\mathcal{H}(\mathbf{KL}_p)$ acts transitively on each level L_i, demonstrating that $d_H(\mathbf{KL}_p) = p + 1$, as desired. □

(*iii*) *Krull Dimension of a Space*

Let X be a topological space. A subset S of X is said to be *irreducible* if, whenever C_1 and C_2 are closed sets in X such that $S \subseteq C_1 \cup C_2$, it follows that $S \subseteq C_1$ or $S \subseteq C_2$. Equivalently, every nonempty open subset of S is dense in S. Following [9], a space X is said to be sober, if every closed irreducible set S of X has a unique generic point (i.e., $S = \overline{\{a\}}$, for a unique $a \in S$).

Let $\mathbf{IC}(X)$ denote the set of all nonempty irreducible closed sets in X. The *Krull dimension of X*, denoted by $\mathrm{Kdim}(X)$, is defined as the height of the poset $(\mathbf{IC}(X), \subseteq)$.

In particular, $\mathrm{Kdim}(X) = -1$ if and only if $X = \emptyset$.

The *Krull dimension* of X at a point $x \in X$ is defined as:

$$\mathrm{Kdim}_x(X) = \min\{\mathrm{Kdim}(U) : x \in U \subseteq X \text{ is open}\}.$$

The following theorem establishes a relationship between the Krull dimension and the height of a T_0-space.

Theorem 5.7. *Let X be a T_0-space. Then the following properties hold:*

1. $\mathrm{ht}(X) \leq \mathrm{Kdim}(X)$.
2. *If X is a sober space, then* $\mathrm{ht}(X) = \mathrm{Kdim}(X)$.
3. *If X is an Alexandroff space, then* $\mathrm{ht}(X) = \mathrm{Kdim}(X)$.

Proof. The case when the space is empty is straightforward. So, one may assume that $X \neq \emptyset$. We denote by $\leq$ the ordering induced by the topology on X; which is defined by

$$x \leq y \text{ if and only if } y \in \overline{\{x\}}.$$

1. Let $x_0 < x_1 < \ldots < x_n$ be a chain of $(X, \leq)$ of length n, then

$$\overline{\{x_n\}} \subset \ldots \subset \overline{\{x_1\}} \subset \overline{\{x_0\}}$$

   is a chain of irreducible closed sets of the space X. Thus $\mathrm{ht}(X) \leq \mathrm{Kdim}(X)$.
2. Assume X is sober, and let $F_0 \subset F_1 \subset \ldots \subset F_n$ be a chain of irreducible closed sets of X; then as X is sober, for every i, there exist an $x_i \in F_i$ such that $F_i = \overline{\{x_i\}}$. This leads to the chain $x_n < \ldots < x_1 < x_0$ of elements of the poset $(X, \leq)$. Therefore $\mathrm{Kdim}(X) \leq \mathrm{ht}(X)$; and consequently $\mathrm{Kdim}(X) = \mathrm{ht}(X)$.
3. If $\mathrm{ht}(X) = +\infty$, then by the first point $\mathrm{Kdim}(X) = \mathrm{ht}(X)$. Now, assume X is of finite height. We will show that X is sober. Indeed, let C be an irreducible closed set of X. Suppose that X has no generic point. We claim that for every $x_0 \in C$, there exists $x_1 \in C$ such that $x_1 < x_0$. Indeed, as $\overline{\{x_0\}} \subset C$, there exists $y_0 \notin \overline{\{x_0\}}$. As C is irreducible, and $(\downarrow x_0] \cap C$, $(\downarrow y_0] \cap C$ are nonempty open sets of C, their intersection is not empty. So there exist $x_1 \in (\downarrow x_0] \cap (\downarrow y_0] \cap C$. Hence $x_1 \in C$ and $x_1 < x_0$. This process allows us to construct a strictly decreasing sequence $\{x_n : n = 0, 1, \ldots\}$, contradicting the fact that X is of finite height.

□

Example 5.8. *For a non-sober space X, it is possible that* $\mathrm{ht}(X) < \mathrm{Kdim}(X)$. *Consider, for instance, an infinite set X equipped with the co-finite topology:*

$$\mathcal{T} = \{O \subseteq X \mid O = \emptyset \textit{ or } X - O \textit{ is finite}\}.$$

It is evident that a closed set C of X is irreducible if and only if either $C = X$ or C is a singleton. Thus, we have $\mathrm{Kdim}(X) = 1$. *However, since every singleton is closed, it follows that* $\mathrm{ht}(X) = 0$.

(*iv*) *Small Inductive Dimension of a Space*

The small inductive dimension of a space X, denoted $\mathrm{ind}(X)$, is a value chosen from the set $\{-1, 0, 1, 2, 3, \ldots, \infty\}$ and is defined recursively as follows:

The empty space $\emptyset$ has $\mathrm{ind}(X) = -1$. Next, for a nonnegative integer k, we say $\mathrm{ind}(X) \leq k$ if and only if there exists a base for the open sets of X consisting of sets U such that $\mathrm{ind}(\partial_X(U)) \leq k-1$, where $\partial_X(U)$ is the boundary of U in the space X. We define $\mathrm{ind}(X) = k$ if $\mathrm{ind}(X) \leq k$ but $\mathrm{ind}(X) \nleq k-1$. Finally, if $\mathrm{ind}(X) \leq k$ is false for all integers k, we set $\mathrm{ind}(X) = \infty$.

The small inductive dimension is also known as the *Urysohn-Menger dimension* or the *weak inductive dimension*.

For Alexandroff T_0-spaces, the small inductive dimension coincides with the height.

Theorem 5.9 ([29]). *Let X be a T_0-Alexandroff space, then we have*

$$\dim(X) = \mathrm{ht}(X) = \mathrm{Kdim}(X).$$

(*v*) *Large Inductive Dimension of a Space*

According to [7], the *large inductive dimension* of a topological space X, denoted by $\mathrm{Ind}(X)$, is defined recursively as follows:

$\text{Ind}(X) = -1$ for $X = \emptyset$. Next, for a nonnegative integer k, $\text{Ind}(X) \leq k$ if for every pair (C, O) of subsets of X, where C is closed, O is open, and $C \subseteq O$, there exists an open set U of X such that $C \subseteq U \subseteq O$ and $\text{Ind}(\partial_X(U)) \leq k-1$. Finally, $\text{Ind}(X) = k$, if $\text{Ind}(X) \leq k$ and $\text{Ind}(X) \nleq k-1$. Finally, if $\text{Ind}(X) \leq k$ is false for all integers k, we set $\text{Ind}(X) = \infty$. The large inductive dimension is also known as the *Čech dimension* or the *strong inductive dimension*.

The following theorem justifies the names of the small and the large inductive dimensions.

Theorem 5.10 ([7]).

1. *For every normal space X we have* $\text{ind}(X) \leq \text{Ind}(X)$.
2. *For every separable metric space X we have* $\text{ind}(X) = \text{Ind}(X)$.

In general, for an Alexandroff T_0-space, the inequality $\text{Ind}(X) \leq \text{ind}(X)$ holds [8, Corollary 5.4].

For a subset A of an Alexandroff space X, we denote by $(\downarrow A]$ the smallest open set of X containing A (the intersection of all open sets containing A).

Proposition 5.11 ([8, Corollary 5.2]). *Let X be an Alexandroff space and k be a nonnegative integer. Then,* $\text{Ind}(X) = k$ *if and only if* $\text{Ind}(X) \leq k$ *and there exists a closed subset C of X such that* $\text{Ind}\,(\partial_X((\downarrow C])) = k-1$.

Unfortunately, we do not yet have an answer regarding the large inductive dimension of $\mathbf{KL}_p$ for $p \geq 2$.

Proposition 5.12. *The large inductive dimension of the digital line* $\mathbf{KL}_1$ *is* 1.

Proof. From [8, Corollary 5.4], we have $\text{Ind}(\mathbf{KL}_1) \leq \text{ind}(\mathbf{KL}_1) = 1$.

To conclude, we will apply Proposition 5.11. Consider the closed set $C = \{2n\}$. We have $(\downarrow C] = \{2n-1, 2n, 2n+1\}$ and $\overline{(\downarrow C]} = \{2(n-1), 2n-1, 2n, 2n+1, 2(n+1)\}$. Thus, the boundary of $(\downarrow C]$ is $\{2(n-1), 2(n+1)\}$, and consequently $\text{Ind}(\partial_{\mathbf{KL}_1}((\downarrow C]) = 0$. This demonstrates that $\text{Ind}(\mathbf{KL}_1) = 1$. □

Question 5.13. *Is* $\text{Ind}(\mathbf{KL}_p) = 1$ *for every* $p \geq 1$*?*

Acknowledgments

The author gratefully acknowledges the support provided by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum and Minerals. The author also sincerely appreciates the valuable remarks, suggestions, and corrections offered by the anonymous referee, which have significantly enhanced the relevance and quality of this paper.

References

[1] P. Alexandroff, *Diskrete Räume.*, Rec. Math. [Mat. Sbornik] N.S **44** (1937), 501–519.
[2] E. Aşı cı, F. Karaçal, *On the T-partial order and properties*, Inform. Sci. **267** (2014), 323–333.
[3] V. A. Chatyrko, S. E. Han, Y. Hattori, *The small inductive dimension of subsets of Alexandroff spaces*, Filomat **30** (2016), 3007–3014.
[4] K. Conrad, *Dihedral Groups II*, available at: `https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf.`
[5] P. H. Doyle, J. G. Hocking, *Continuous bijections on manifolds*, J. Austral. Math. Soc. Ser. A **22** (1976), 257–263.
[6] G. Edgar, *Measure, topology, and fractal geometry*, Undergrad. Texts Math., Springer, New York (2008).
[7] R. Engelking, *Theory of dimensions finite and infinite*, Sigma Series in Pure Mathematics, 10. Heldermann Verlag, Lemgo, 1995.
[8] D. N. Georgiou, A. C. Megaritis, S. P. Moshokoa, *Small inductive dimension and Alexandroff topological spaces*, Topology Appl. **168** (2014), 103–119.
[9] A. Grothendieck, J. A. Dieudonné, *Éléments de géométrie algébrique. I*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971.
[10] S. E. Han, *Adjacency relations induced by some Alexandroff topologies on* $\mathbb{Z}^n$, AIMS Math. **7** (2022), 11581–11596.
[11] S. E. Han, S. Jafari, J. M. Kang, S. Lee, *Remarks on topological spaces on* $\mathbb{Z}^n$ *which are related to the Khalimsky n-dimensional space*, AIMS Math. **7** (2022), 1224–1240.
[12] S. E. Han, J. Lee, W. Yao, J. Kim, *Existence of a proper subspace of* $(\mathbb{Z}^n, ((T_{S_k}))^n)$ *which is homeomorphic to the n-dimensional Khalimsky topological space*, Topology Appl. **344** (2024), Paper No. 108812, 17 pp.
[13] G. T. Herman, *Geometry of digital spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 1998.
[14] J. G. Hocking, P. H. Doyle, *Strongly reversible manifolds*, J. Austral. Math. Soc. Ser. A **34** (1983), 172–176.

[15] B. Honari, Y. Bahrampour, *Cut-point spaces*, Proc. Amer. Math. Soc. **127** (1999), 2797–2803.
[16] F. Karaçal, T. Köroğlu, *A principal topology obtained from uninorms*, Kybernetika (Prague) **58** (2022), 863–882.
[17] E. D. Khalimsky, *On topologies of generalized segments*, Soviet Math. Dokl., **10** (1969), 1508–1511.
[18] E. Khalimsky, *Topological structures in computer science*, J. Appl. Math. Simulation **1** (1987), 25–40.
[19] E. Khalimsky, R. Kopperman, P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. **36** (1990), 1–17.
[20] C. O. Kiselman, *Elements of digital geometry, mathematical morphology, and discrete optimization*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2022.
[21] S. Lazaar, T. Richmond, H. Sabri, *The autohomeomorphism group of connected homogeneous functionally Alexandroff spaces*, Comm. Algebra **47** (2019), 3818–3829.
[22] E. Melin, *Extension of continuous functions in digital spaces with the Khalimsky topology*, Topology Appl. **153** (2005), 52–65.
[23] E. Melin, *Digital Khalimsky manifolds*, J. Math. Imaging Vision **33** (2009), 267–280.
[24] M. Rajagopalan, A. Wilansky, *Reversible topological spaces*, J. Austral. Math. Soc. **6** (1966), 129–138.
[25] J. Rotman, *An introduction to the theory of groups*, Grad. Texts in Math., 148 Springer-Verlag, New York, 1995.
[26] A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. **122** (1966), 379–398.
[27] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.
[28] J. V. Whittaker, *On isomorphic groups and homeomorphic spaces*, Ann. of Math. (2) **78** (1963), 74–91.
[29] P. Wiederhold, R. G. Wilson, Dimension for Alexandroff spaces, in: R.A. Melter, A.Y. Wu (Eds.), Vision Geometry, in: Proc. Soc. Photo-Opt. Instrum. Eng. (SPIE), vol. 1832, 1993, pp. 13–22.