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Introduction to potential theory in the class of m-convex functions

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Abstract. In this work we will give the very initial concepts of potential theory ($m-cv$ polar sets, $m-cv$ measures and their properties) in the class of m-convex ($m-cv$) functions in the domain $D \subset \mathbb{R}^n_x$. In particular, we will prove that for a $m-cv$ measure $\omega^*(x, E, D)$ its Hessian $H^{n-m+1}_{\omega^*} = 0$ in the domain $D \backslash E$.

1. Introduction

In this work we will give the very initial concepts of potential theory ($m-cv$ polar sets, $m-cv$ measures and their properties) in the class of m-convex ($m-cv$) functions in the domain $D \subset \mathbb{R}^n_x$. In particular, we will prove that for a $m-cv$ measure $\omega^*(x, E, D)$ its Hessian $H^{n-m+1}_{\omega^*} = 0$ in the domain $D \backslash E$.

If the potential theory in the class of strongly m-subharmonic functions is based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geq 0,\ \ k = 1, 2, ..., n-m+1$, where $\beta = dd^c \|z\|^2$ the standard volume form in $\mathbb{C}^n$, then the potential theory in the class of $m-cv$ functions is based on Borel measures of a completely different nature, namely, on Hessians $H^k(u) \geq 0,\ \ k = 1, 2, ..., n-m+1$. In the work of A. Sadullaev [9] (see also the work of R. Sharipov and M. Ismoilov [8]) it was proved that in the class of bounded m-convex functions the Hessians $H^k(u) \geq 0,\ \ k = 1, 2, ..., n-m+1$, are defined and are positive Borel measures. Note that when $m = n$ the class $m-cv$ coincides with the class of subharmonic functions, and when $m = 1$ it coincides with the class of convex functions. The classes of subharmonic and convex functions are well studied (see [6], [10], [1]-[2], [3], [4]–[5]).

2. $m-cv$ polar sets.

Definition 2.1. *By analogy with polar sets in classical potential theory, a set $E \subset D \subset \mathbb{R}^n$ is called $m-cv$ polar in D, if there exists a function $u(x) \in m-cv(D)$, $u(x) \not\equiv -\infty$, such that $u|_E = -\infty$.*

From the embedding $m-cv(D) \subset sh(D)$ it follows that every $m-cv$ polar set is polar in the sense of classical potential theory. In particular, for a $m-cv$ polar set E it is true $H_{2n-2+\varepsilon}(E) = 0, \forall \varepsilon > 0$ and, therefore, the Lebesgue measure of a $m-cv$ polar set E is equal to zero.

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$m-cv$ polar sets have another unexpected phenomenon, that when $m<\frac{n}{2}+1$ they are empty, i.e. if the set $E\subset D$ is $m-cv$ polar, $m<\frac{n}{2}+1$, then $E=\emptyset$. This follows from the fact that for $m<\frac{n}{2}+1$ any $m-cv$ function is continuous. (see [11]-[13], [14], [8]). However, for $m\geq\frac{n}{2}+1$ there are non-empty $m-cv$ polar sets. Therefore, the properties of $m-cv$ polar sets proved below are meaningful only for the cases $m\geq\frac{n}{2}+1$.

Example 2.2. *(fundamental $m-cv$ function).*

$$\chi_m(x,0)=\begin{cases}|x|^{2-\frac{n}{n-m+1}} & at \quad m<\frac{n}{2}+1\\ \ln|x| & at \quad m=\frac{n}{2}+1\\ -|x|^{2-\frac{n}{n-m+1}} & at \quad m>\frac{n}{2}+1\end{cases} \tag{1}$$

Thus, at the $m<\frac{n}{2}+1$ fundamental function is bounded and Lipschitz, and at $m\geq\frac{n}{2}+1$ it is equal $-\infty$ at the point $x=0$ Note that at $m=n$, i.e. for the subharmonic case it coincides with the fundamental solution $-\frac{1}{|x|^{n-2}}$ of the Laplace operator Δ.

Theorem 2.3. *The countable union of $m-cv$ polar sets is $m-cv$ polar, i.e. if $E_j\subset D$ is $m-cv$ polar, then $E=\bigcup_{j=1}^{\infty}E_j$ is also $m-cv$ polar.*

T h e p r o o f is identical to a similar proof for a polar sets and we omit it.

Potential theory is usually constructed in regular domains with respect to one or another class of functions.

Definition 2.4. *A domain $D\subset\mathbb{R}^n$ is called $m-cv$ regular if there exists $\rho(x)\in m-cv(D)$ such that $\rho(x)<0$, $\varliminf_{x\to\partial D}\rho(x)=0$. It is called $m-cv$ strictly regular if there is a twice smooth strictly $m-cv$ function in some neighborhood $D^+\supset\bar{D}$ of the closure $\bar{D}$ such that $D=\{\rho(x)<0\}$. Strictly $m-cv$ of a function $\rho(x)$ in D^+ means that for some $\delta>0$ the difference $\rho(x)-\delta\|x\|^2$ is a $m-cv$ function in D^+.*

Theorem 2.5. *Let a domain $D\subset\mathbb{R}^n$ be $m-cv$ regular and the subset $E\subset D$ such that the intersection $E\cap G$ is $m-cv$ polar in G for an arbitrary compact subdomain $G\subset\subset D$. Then E is $m-cv$ polar in D. Moreover, there is a function $u(x)\in m-cv(D)$, $u|_D<0:\quad u(x)\not\equiv-\infty,\ u(x)=-\infty\ \forall x\in E$.*

P r o o f. The theorem is very useful in proving the more general result that a locally $m-cv$ polar set is a globally (overall $\mathbb{R}^n$) $m-cv$ polar set. Since $D\subset\mathbb{R}^n$ is $m-cv$ regular, then there exists $\rho(x)\in m-cv(D)$ such that $\rho(x)<0$, $\varliminf_{x\to\partial D}\rho(x)=0$. We put $D_\delta=\{x\in\partial D:\rho(x)<-\delta\}\subset\subset D,\ \delta>0$. Using the connected components of the open sets D_δ, we construct the exhaustion $G_j\subset\subset G_{j+1},\ \bigcup_{j=1}^{\infty}G_j=D,\ G_1\neq\emptyset$, where G_j is suitable connected component of the open set $D_{\delta_j}=\left\{x\in\partial D:\rho(x)<-\delta_j\right\}\subset\subset D,\ \delta_j>0,\ \delta_j\downarrow 0$.

According to the conditions of the theorem, there is a function $v_j(x)\in m-cv(G_{j+2})$ such that $v_j\not\equiv-\infty$, but $v_j\big|_{E\cap G_{j+2}}\equiv-\infty$. Since the set $\{v_j=-\infty\}$ has Lebesgue measure zero, there is a point $a\in G_1$ such that $v_j(a)\neq-\infty$ for all $j\in N$.

Let's put $M_j=\max_{x\in\bar{G}_{j+1}}v_j(x)$, $V_j(x)=\frac{1}{2^j}\cdot\frac{v_j(x)-M_j}{M_j-v_j(a)}$ and $u_j(x)=A_j(\rho(x)+\delta_{j+1})$, where $A_j>0$ is that $u_j\big|_{G_j}\leq-1$. Then $V_j(x)\big|_{G_{j+1}}<0$, $u_j\big|_{\partial G_{j+1}}\equiv 0$ and, therefore, the function

$$w_j(x)=\begin{cases}\max\left\{V_j(x),u_j(x)\right\}, & at \quad x\in G_{j+1}\\ u_j(x), & at \quad x\notin G_{j+1}\end{cases}$$

is $m-cv$ in D $(j=1,2,...)$, $w_j(x)<0$ in G_{j+1}.

The sum $w(x)=\sum_{j=1}^{\infty}w_j(x)\in m-cv(D)$, $w(a)=-1, w|_E\equiv-\infty$. It follows that E is $m-cv$ polar in D. Note that if we select in advance a sequence δ_j converging to zero quickly, for example as $\delta_j=\frac{1}{(j!)^2}$, then we will get $w(x)$ bounded in D, $w(x)\leq C$. *The theorem is proven.*

3. $m - cv$ measure

In the theory of m-convex functions, the $m - cv$ measure plays the same role as the harmonic measure in the classical potential theory. To exclude trivial cases, regular or even strictly $m - cv$ regular domains $D \subset \mathbb{R}^n$ are usually considered as a fixed domain.

Let $E \subset D$ be some subset of a strictly $m - cv$ regular domain $D \subset \mathbb{R}^n$.

Definition 3.1. *Consider the class of functions*

$$\mathcal{U}(E, D) = \{u(x) \in m - cv(D) : \quad u|_D \leq 0, \quad u|_E \leq -1\} \tag{2}$$

and put $\omega(x, E, D) = \sup\{u(x) : \ u \in \mathcal{U}(E, D)\}$. Then the regularization $\omega^*(x, E, D)$ is called the $m - cv$ measure of the set E with respect to the domain D.

From the property of the upper envelope of $m - cv$ functions (see [7]) it follows that $\omega^*(x, E, D) \in m - cv(D)$. By Choquet's lemma (see [6], [10]) there exists a countable subfamily $\mathcal{U}' \subset \mathcal{U}(E, D)$ such that $\{\sup\{u(x)\} : \ u \in \mathcal{U}'(E, D)\}^* \equiv \omega^*(x, E, D)$. It follows that the $m - cv$ measure $\omega^*(x, E, D)$ can be represented as the limit of a monotonically increasing sequence $\left\{u_j(x)\right\} \subset \mathcal{U}(E, D)$: $\left[\lim\limits_{j\to\infty} u_j(x)\right]^* \equiv \omega^*(x, E, D)$.

In the special case, when $E \subset D$ is compact, the functions $u_j(x) \subset \mathcal{U}(E, D)$ can be selected to be continuous in D, which can be easily verified by convexly continuing $u_j(x) \subset \mathcal{U}(E, D)$ into a certain fixed neighborhood $D^+ \supset \bar{D}$ and approximating u_j with smooth functions $u_k = u \circ K_k(x - y)$.

Properties of the $m - cv$ measure:

1) (monotonicity) if $E_1 \subset E_2$, then $\omega^*(x, E_1, D) \geq \omega^*(x, E_2, D)$; if $E \subset D_1 \subset D_2$, then $\omega^*(x, E, D_1) \geq \omega^*(x, E, D_2)$;

2) $\omega^*(x, U, D) \in \mathcal{U}(U, D)$ for open set $U \subset D$ and, therefore $\omega^*(x, U, D) \equiv \omega(x, U, D)$;

It follows from the fact that for concentric balls $B(x^0, r) \subset B(x^0, R) \subset\subset U$, $0 < r < R$, the $m - cv$ measure

$$\omega^*(x, \bar{B}(x^0, r), B(x^0, R)) = \max\left\{-1, \ \frac{\chi_m(x, x^0) - \chi_m(R, 0)}{\chi_m(R, 0) - \chi_m(r, 0)}\right\},$$

and therefore in both cases $m < \frac{n}{2} + 1$ or $m \geq \frac{n}{2} + 1$ we have $\omega^*(x^0, U, D) = -1$. Here $\chi_m(x, x^0)$ is a fundamental $m - cv$ function (see (1)).

3) if $U \subset D$ is an open set, $U = \bigcup\limits_{j=1}^{\infty} K_j$, where $K_j \subset \overset{\circ}{K}_{j+1}$, then $\omega^*(x, K_j, D) \downarrow \omega(x, U, D)$ (*easily follows from property* 2).

4) if $E \subset D$ an arbitrary set, then there is a decreasing sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$ $(j = 1, 2, ...)$, such that $\omega^*(x, E, D) = \left[\lim\limits_{j\to\infty} \omega(x, U_j, D)\right]^*$.

In fact, if the sequence $\left\{u_j(x)\right\} \subset \mathcal{U}(E, D)$ is monotonically increasing: $\left[\lim\limits_{j\to\infty} u_j(x)\right]^* \equiv \omega^*(x, E, D)$, then the open set $U_j = \left\{u_j < -1 + \frac{1}{j}\right\}$ has the property, that $U_j \supset E$, $U_j \supset U_{j+1}$ $(j = 1, 2, ...)$. Hence, $u_j(x) - \frac{1}{j} < -1$ $\forall x \in U_j$ and $u_j(x) - \frac{1}{j} \leq \omega^*(x, U_j, D) \leq \ \omega^*(x, E, D)$ $\forall x \in D$. Therefore, from $\left[\lim\limits_{j\to\infty} u_j(x)\right]^* \equiv \omega^*(x, E, D)$ we get $\omega^*(x, E, D) = \left[\lim\limits_{j\to\infty} \omega(x, U_j, D)\right]^*$

5) the $m - cv$ measure $\omega^*(x, E, D)$ is either nowhere equal to zero or identically equal to zero. $\omega^*(x, E, D) \equiv 0$ if and only if E is $m - cv$ polar in D.

Remark 3.2. *Property 5 is meaningful only for $m \geq \frac{n}{2} + 1$ when $m < \frac{n}{2} + 1$ a non-empty $m - cv$ polar set does not exist, so the trivial $m - cv$ measure does not exist either.*

Example 3.3. *Consider $m = 1$, take a ball $B = B(0,1)$ and a set $E = \{0\}$ in it, consisting only one point. Consider the $1-cv$ measure $v = \omega^*(x,E,D),\ x \in \mathbb{R}^n,\ v \in \mathbb{R}$, as a function in $\mathbb{R}^{n+1}_{(x,v)}$. Then it is easy to see that the convex function $v = \omega^*(x,E,D),\ x \in \mathbb{R}^n,\ v \in \mathbb{R}$, will be a cone, with a vertex at a point $(0,-1)$ and with the base $\{x \in \partial B,\ v = 0\}$. Thus, the $1-cv$ measure $\omega^*(x,E,D) \neq 0$.*

P r o o f o f P r o p e r t y 5. The first part of the property follows from the maximum principle for $m-cv$ functions: if $\omega^*(x^0,E,D) = 0$ at some point $x^0 \in D$, then $\omega^*(x,E,D) \equiv 0$ in D.

Let $\omega^*(x,E,D) \equiv 0$ be now. Then there exists a point $x^0 \in D$ such that $\omega^*(x^0,E,D) = 0$. By definition ω, for each $j \in N$ there is a function $u_j(x) \subset \mathcal{U}(E,D)$ such that $u_j(x^0) \geq -\frac{1}{2^j}$. Let's consider the sum $u(x) = \sum\limits_{j=1}^{\infty} u_j(x)$. It is m-convex because $u_j < 0$ in D_j; $u(x) \not\equiv -\infty$, because $u(x^0) \geq -1$, $u|_E \equiv -\infty$ because $u_j\big|_E \leq -1$. This shows that E is a $m-cv$ polar set in D.

And vice versa, for a $m-cv$ polar set $E \subset D$, according to Theorem 2.3, there is a function $v(x) \in m-cv(D):\ v|_D < 0,\ v(x) \not\equiv -\infty$ but $v|_E \equiv -\infty$. Then the function $\frac{v(x)}{j} \in \mathcal{U}(E,D)$ for any $j \in N$. Consequently $\omega(x,E,D) \geq \frac{v(x)}{j}$, we also get from here that: $\omega(x,E,D) = 0\ \ \forall x \in D:\ v(x) \neq -\infty$. Since the Lebesgue measure of the set $\{v = -\infty\}$ is equal to zero, then $\omega^*(x,E,D) \equiv 0$ in D.

6) (*theorem on two constants*). If the function $u(x)$ is m-convex in $D \subset \mathbb{R}^n$ and $u|_D \leq M,\ \ u|_E \leq m,\ \ (E \subset D)$, then for all $x \in D$ the following inequality holds

$$u(x) \leq M(1 + \omega^*(x,E,D)) - m\omega^*(x,E,D). \tag{3}$$

F o l l o w i n g out of $\frac{u(x)-M}{M-m} \in \mathcal{U}(E,D)$.

Definition 3.4. *A point $x^0 \in K$ is called a $m-cv$ regular point of a compact K (with respect to D) if $\omega^*(x^0,K,D) = -1$. A compact $K \subset D$ is called a $m-cv$ regular compact if each of its point $x^0 \in K$ is $m-cv$ regular.*

Since $m-cv(D) \subset sh(D)$, then the $m-cv$ measure of a pair (K,D) is always no greater than the harmonic measure of this pair. Consequently, regular compacta in the sense of classical potential theory are always $m-cv$ regular. Therefore, the closure of a domain $G \subset\subset D$ with a doubly smooth boundary ∂G is a $m-cv$ regular compact. It follows that for any compact set $K \subset U \subset D$, where U is an open set, there is always a pluriregular compact set $F:\ K \subset F \subset\subset U \subset D$. This follows from the fact that for $K \subset U$ it is easy to construct an open set $G \subset\subset U,\ G \supset K$, with a twice piecewise smooth boundary ∂G. Therefore, the compact $F = \bar{G}$ is the desired $m-cv$ regular compact. All this shows that the family of $m-cv$ regular compacta is quite rich.

7) if the set E lies compactly in a strictly $m-cv$ regular domain, $D = \{\rho(x) < 0\},\ E \subset\subset D$ then the measure $m-cv$ extends to a neighborhood $\rho(x) < \delta,\ \delta > 0$, of the closure $\bar{D}$.

I n f a c t, since E is a compact set, then there is a constant $C > 0$ such that $C\rho(x) < -1\ \ \forall x \in E$. It follows that $C\rho(x) \in \mathcal{U}(E,D)$ and $C\rho(x) \leq \omega^*(x,E,D)$. Therefore, the function

$$w(x) = \begin{cases} \max\{C\rho(x), \omega^*(x,E,D)\} & at\ \ x \in D \\ C\rho(x) & at\ \ x \notin D \end{cases}$$

is $m-cv$ in some neighborhood $D^+ \supset \bar{D}$, $w(x) = \omega^*(x,E,D)\ \ \forall x \in D$.

The following theorem plays an important role in introducing the condenser capacity and in further studying the potential properties of $m-cv$ convex functions.

Theorem 3.5. *If a compact set $E \subset D$ is $m-cv$ regular, then the $m-cv$ measure $\omega^*(x,E,D) \equiv \omega(x,E,D)$ and is a continuous function in D, $\omega^*(x,E,D) \in C(D)$.*

P r o o f. According to property 7), we extend the $m-cv$ measure $\omega^*(x,E,D)$ into a neighborhood $\rho(x) < \delta,\ \delta > 0$, of the closure $\bar{D}$ and approximate it in a certain neighborhood

$$D^+ \supset \bar{D}:\ \ u_j(x) \downarrow \omega^*(x,E,D),\ x \in D^+,\ \ u_j(x) \in C^\infty(D^+) \cap m-cv(D^+).$$

We fix a number $\varepsilon > 0$. Applying Hartogs' Lemma twice to the sequence $u_j(x) \downarrow \omega^*(x, E, D)$ and to neighborhoods $U = \{\omega^*(x, E, D) < -1 + \varepsilon\} \supset E$, $\check{D} = \{\omega^*(x, E, D) < \varepsilon\} \supset \bar{D}$, we find a number $j_0 \in \mathbb{N}$: $u_j(x) < -1 + 2\varepsilon$, $\forall x \in E$, $u_j(x) < 2\varepsilon$, $\forall x \in \bar{D}$, $j \geq j_0$. Then $u_j(x) - 2\varepsilon < -1, \forall x \in E, u_j(x) - 2\varepsilon < 0$, $\forall x \in D$, $j \geq j_0$, i.e. $u_j(x) - 2\varepsilon < -1, \forall x \in E, u_j(x) - 2\varepsilon < 0, \forall x \in D$, $j \geq j_0$ and $u_j(x) - 2\varepsilon \in \mathcal{U}(E, D), \omega^*(x, E, D) - 2\varepsilon \leq u_j(x) - 2\varepsilon \leq \omega^*(x, E, D)$. This means that the sequence of smooth functions $u_j(x) \downarrow \omega^*(x, E, D)$ converges uniformly and $\omega^*(x, E, D) \in C(D)$. *The theorem is proven.*

4. Properties of Hessians of measures

$m - cv$ measure $\omega^*(x, E, D)$ has the maximality property outside a compact set E (in the domain $D \backslash E$). This means that for any domain $G \subset\subset D \backslash E$ the inequality $\omega^*(x, E, D) \geq v(x)$ $\forall x \in G$ holds for all functions $v \in m - cv(D)$: $\omega^*(x, E, D)|_{\partial G} \geq v(x)|_{\partial G}$. To prove the maximality property, we first recall the definition of Hessians H^k, $k = 1, 2, ..., n - m + 1$, for a bounded semi-continuous function $u(x) \in m - cv(D) \cap L^\infty(D)$. They are positive Borel measures (see [9]). Let us embed $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y (z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n_z$. Then an upper semi-continuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ will be m-convex if and only if the function $u^c(z) = u^c(x + iy) = u(x)$ that does not depend on variables $y \in \mathbb{R}^n_y$, is strongly $m-$subharmonic ($u^c \in sh_m$) in the domain $D \times \mathbb{R}^n_y$ (see [7], [9]).

If m-convex in a domain $D \subset \mathbb{R}^n_x$ function $u(x) \in m - cv(D)$ is bounded, then $u^c(z)$ will also be a bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$. As it is well known, the operators $(dd^c u^c)^k \wedge \beta^{n-k}$, $k = 1, 2, ..., n - m + 1$ are defined for a bounded sh_m functions as Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, $\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a doubly smooth function,

$$(dd^c u^c)^k \wedge \beta^{n-k} = k!(n - k)!H^k(u^c)\beta^n,$$

then for a bounded, strongly subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, it is natural to determine its Hessians, equating them to the measure

$$H^k(u^c) = \frac{\mu_k}{k!(n - k)!} = \frac{1}{k!(n - k)!} (dd^c u^c)^k \wedge \beta^{n-k}, \tag{4}$$

$$\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}, \ \ k = 1, 2, ..., n - m + 1.$$

Since $u^c \in sh_m \left(D \times \mathbb{R}^n_y\right)$ does not depend on $y \in \mathbb{R}^n_y$, then for any Borel sets $E_x \subset D$, $E_y \subset \mathbb{R}^n_y$ the measures $\frac{1}{mesE_y}\mu_k(E_x \times E_y) = \nu_k(E_x)$ do not depend on the set $E_y \subset \mathbb{R}^n_y$, i.e. $\frac{1}{mesE_y}\mu_k(E_x \times E_y) = \nu_k(E_x)$. We will call Borel measures ν_k : $\nu_k(E_x) = \frac{1}{mesE_y}\mu_k(E_x \times E_y)$, $k = 1, 2, ..., n - m + 1$, as Hessians H^k, $k = 1, 2, ..., n - m + 1$ for a bounded, m-convex function $u(x) \in m - cv(D)$ in the domain $D \times \mathbb{R}^n_x$. For a doubly smooth function $u(x) \in m - cv(D) \cap C^2(D)$, the Hessians are ordinary functions; however, for a non-doubly smooth, but bounded semicontinuous function $u(x) \in m - cv(D) \cap L^\infty(D)$, the Hessians H^k, $k = 1, 2, ..., n - m + 1$, are positive Borel measures.

Theorem 4.1. *For any $m - cv$ regular compact set $E \subset D$, the $m - cv$ measure $\omega^*(x, E, D)$ is a maximal function in $D \backslash E$. Moreover, $H^{n-m+1}_{\omega^*} = 0$ in the domain $D \backslash E$.*

P r o o f. a) The $m - cv$ measure $\omega^*(x, E, D) \equiv \omega(x, E, D)$ is maximal in $D \backslash E$. In fact, if $\omega(x, E, D)$ is not maximal function in $D \backslash E$, then there is a domain $G \subset\subset D$ and a function $\phi(x) \in m - cv(D)$: $\phi\big|_{\partial G} \leq \omega|_{\partial G}$, but $\phi(x^0) > \omega(x^0)$ at some point $x^0 \in G$. Then the function

$$w(x) = \begin{cases} \max\left\{\omega(x), \phi(x)\right\} & at \ \ x \in \bar{G} \\ \omega(x) & at \ \ x \in D \backslash G \end{cases}$$

is m-convex, $w(x) \in m - cv(D)$, $w(x)|_D < 0$, $w(x)|_E \leq -1$ because $w(x) = \omega(x)$ $\forall x \in D \backslash E$. Hence, $w(x) \leq \omega(x)$ and $\phi(x^0) \leq \omega(x^0)$. Contradiction.

b) $H^{n-m+1}_{\omega^*} = 0$ in the $D\backslash E$. R e a l l y, it is not difficult to see that the function $u(x) \in m - cv(G) \cap C(D)$ is maximal if and only if the function $u^c(z) \in sh_m\left(G \times \mathbb{R}^n_y\right) \cap C\left(G \times \mathbb{R}^n_y\right)$ is maximal sh_m function. This implies $(dd^c u^c)^{n-m+1} \wedge \beta^{m-1}$, which is the same $H^{n-m+1}(u^c) = 0$. This is equivalent to the fact that $H^{n-m+1}(u(x)) = 0$. *The theorem is proven.*

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