Filomat 39:9 (2025), 3035–3042
https://doi.org/10.2298/FIL2509035A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a Golden deformation of paracontact structures

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Abstract. The main purpose of this article is to define the concept of Golden deformation via the relationship between Golden structure and paracontact structure on the almost paracontact para-Norden manifold. The Riemannian connection is obtained on the Golden-deformed para-Sasaki-like para-Norden manifold. After obtaining the Riemannian connection, the curvature tensor, scalar curvature, $*$-scalar curvature and Ricci tensor are obtained. Finally, an example is provided on the Golden-deformed 5-dimensional para-Sasaki-like para-Norden manifold.

1. Introduction

Manifolds endowed with specific differential-geometric structures have been extensively researched in differantial geometry. Several authors have delved extensively researched the study of almost paracontact structures on manifolds. The notion of almost paracontact paracomplex Riemannian manifold was introduced in the studies by Manev and collaborators, specifically in [11]. Various topics have been addressed on almost paracontact paracomplex Riemannian manifold [3, 10, 11].

The Golden Ratio has held a significant position since ancient times in various fields such as geometry, architecture, music, art, and philosophy. The Golden ratio σ is the positive root of the polynomial $x^2 - x - 1 = 0$; i.e, $\sigma = \frac{1+\sqrt{5}}{2}$.The negative root of the previous equation, usually denoted by σ^*, satisfies $\sigma^* = \frac{1-\sqrt{5}}{2} = 1 - \sigma$. Crasmareanu and Hretcanu introduced and analyzed Golden structures like almost product, almost complex structures and almost contact structures on a differentiable manifold in [4]. The Golden structure is defined by a (1,1)-tensor field Φ on manifold, with the crucial property that $\Phi^2 = \Phi + I$. Recently, researchers have established a relationship between Golden structures and some important geometric structures and have conducted various studies [1, 5–7, 12, 17].

Some studies have been conducted on para-Sasakian manifold and almost contact metric manifold, based on the relationship between Golden structure and almost contact structure [8, 9].

The master goal of this study is to define the notion of the Golden deformation through the relationship between the Golden structure and paracontact structure on almost paracontact para-Norden manifold

2020 *Mathematics Subject Classification*. Primary 58A05; Secondary 58D17.
Keywords. Para-Sasaki-like para-Norden manifold, almost paracontact structure, Golden deformation.
Received: 19 April 2024; Revised: 06 January 2025; Accepted: 21 January 2025
Communicated by Ljubiša D. R. Kočinac
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(referred to as almost paracontact paracomplex Riemannian manifolds in [3, 10, 11]). The progress of the study has been structured as follows. In preliminaries, some fundamental information for almost paracontact para-Norden manifolds is given. Then, the Golden deformation is defined through the relationship between the Golden structure and paracontact structure on almost paracontact para-Norden manifold. The Riemannian connection $\widetilde{\nabla}$, the curvature tensor $\widetilde{R}$, Ricci curvature $\widetilde{Ric}$, scalar curvature $\widetilde{Scal}$ and $*-$scalar curvature $\widetilde{Scal}^*$ are computed on Golden-deformed para-Sasaki-like para-Norden manifolds. Finally, an example is given on a Golden- deformed 5-dimensional para-Sasaki-like para-Norden manifold.

2. Preliminaries

A $(2n+1)-$dimensional differentiable manifold $\left(M, \phi, \xi, \eta\right)$ is referred to as an almost paracontact paracomplex manifold if it is endowed with an almost paracontact structure $\left(\phi, \xi, \eta\right)$, comprising a $(1,1)$-tensor field Φ, a Reeb vector field ξ and its dual 1-form η. The almost paracontact structure $\left(\phi, \xi, \eta\right)$ satisfies the following conditions:

$$\begin{array}{lll} \phi^2 = I - \eta \otimes \xi, & \phi\xi = 0, & \\ \eta(\xi) = 1, & \eta \circ \phi = 0, & tr\,\phi = 0, \end{array} \tag{1}$$

where I is the identity transformation on the tangent bundle TM. $\left(M, \phi, \xi, \eta, g\right)$ is called an almost paracontact para-Norden manifold equipped with a para-Norden metric g relative to $\left(\phi, \xi, \eta\right)$ determined by

$$g\left(\phi x, w\right) = g\left(x, \phi w\right) \tag{2}$$

or equivalently

$$g\left(\phi x, \phi w\right) = g(x, w) - \eta(x)\eta(w) \tag{3}$$

for any smooth vector fields x, w on M, i.e. $x, w \in \chi(M)$ [11, 16]. The almost paracontact para-Norden manifold is briefly called apcpN manifold. As a result, the following equations are obtained:

$$g(x, \xi) = \eta(x), \qquad g(\xi, \xi) = 1, \qquad \eta(\nabla_x \xi) = 0, \tag{4}$$

where ∇ denotes the Riemannian connection of g. From here onwards, x, w, z are arbitrary vector fields from $\chi(M)$ or vectors in TM at a fixed point of M. The basis $\left\{e_0 = \xi,\ e_1, ..., e_n,\ e_{n+1} = \phi e_1, ...,\ e_{2n} = \phi e_n\right\}$ is an orthonormal basis on the structure $\left(\phi, \xi, \eta, g\right)$ with

$$g\left(e_i, e_j\right) = \delta_{ij}, \qquad i, j = 0, 1, ..., 2n. \tag{5}$$

The metric $\widehat{g}$ is an associated metric of g and defined on $\left(M, \phi, \xi, \eta, g\right)$ by

$$\widehat{g}(x, w) = g\left(x, \phi w\right) + \eta(x)\eta(w). \tag{6}$$

The associated metric $\widehat{g}$ is an indefinite metric with signature $(n+1, n)$ and compatible with $\left(M, \phi, \xi, \eta, g\right)$ in a manner analogous to that of g. The apcpN manifolds are classified in [11]. This classification comprises eleven fundamental classes denoted as $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{11}$. The eleven fundamental classes are based on the $(0,3)$-tensor field F determined by

$$F(x, w, z) = g\left(\left(\nabla_x \phi\right) w, z\right) \tag{7}$$

and has the properties

$$F(x, w, z) = F(x, z, w) = -F\left(x, \phi w, \phi z\right) + \eta(w) F(x, \xi, z) + \eta(z) F(x, w, \xi), \tag{8}$$

$$(\nabla_x \eta)(w) = g(\nabla_x \xi, w) = -F\left(x, \phi w, \xi\right).$$

3. Golden deformation

Let $\left(M,\phi,\xi,\eta,g\right)$ be an almost paracontact para-Norden manifold (apcpN manifold). We construct a Golden structure on an apcpN manifold M.

Proposition 3.1. *The* $(1,1)$ *−tensor field* Φ *defined by*

$$\Phi=\frac{1}{2}I+\frac{\sqrt{5}}{2}\left(\phi+\eta\otimes\xi\right) \tag{9}$$

is a Golden structure on an apcpN manifold.

Proof. (9) is written for $x\in\chi(M)$ as

$$\Phi x=\frac{1}{2}x+\frac{\sqrt{5}}{2}\left(\phi x+\eta(x)\xi\right)$$

In order for Φ to be a Golden structure on apcpN manifold, it must satisfy the equation $\Phi^2x=\Phi x+x$ [4]. By utilizing (1) and (9), the equation

$$\Phi^2x=\frac{1}{2}\Phi x+\frac{\sqrt{5}}{2}\left(\Phi\left(\phi x\right)+\eta(x)\Phi\xi\right)$$

is written. Considering the equation $\sigma^2=\sigma+1$ for the eigenvalue σ, the equality $\Phi^2x=\Phi x+x$ is obtained for every $x\in\chi(M)$. Hence, the proof is concluded. □

Proposition 3.2. *Let* $\left(M,\phi,\xi,\eta,g\right)$ *be an apcpN manifold and* Φ *is given as* (9). *In this way, the following equality is satisfied*

$$g(\Phi x,\Phi w)=\frac{3}{2}g(x,w)+\frac{\sqrt{5}}{2}\widehat{g}(x,w). \tag{10}$$

Proof. By utilizing (9), the equality

$$\begin{aligned}g(\Phi x,\Phi w) &= g\left(\frac{1}{2}x,\frac{1}{2}w\right)+g\left(\frac{1}{2}x,\frac{\sqrt{5}}{2}\left(\phi w+\eta(w)\xi\right)\right)\\&\quad+g\left(\frac{\sqrt{5}}{2}\left(\phi x+\eta(x)\xi\right),\frac{1}{2}w\right)\\&\quad+g\left(\frac{\sqrt{5}}{2}\left(\phi x+\eta(x)\xi\right),\frac{\sqrt{5}}{2}\left(\phi w+\eta(w)\xi\right)\right)\end{aligned}$$

is written. Considering (2), (3), (6), (9) and $\sigma+\sigma_*=1$, the equality (10) is obtained. □

Considering Proposition 3.1 and Proposition 3.2, a change in the structures tensors can be generated in form

$$\begin{aligned}\widetilde{\phi} &= \phi,\qquad \widetilde{\xi}=-\frac{1-\sqrt{5}}{2}\xi,\qquad \widetilde{\eta}=\frac{1+\sqrt{5}}{2}\eta, \\ \widetilde{g}(x,w) &= g(\Phi x,\Phi w)=\frac{3}{2}g(x,w)+\frac{\sqrt{5}}{2}\widehat{g}(x,w)\\ &= \frac{3}{2}g(x,w)+\frac{\sqrt{5}}{2}g\left(\phi x,w\right)+\frac{\sqrt{5}}{2}\eta(x)\eta(w).\end{aligned} \tag{11}$$

This is called a Golden deformation on the apcpN manifold. Thus, $\left(M,\widetilde{\phi},\widetilde{\xi},\widetilde{\eta},\widetilde{g}\right)$ is also an apcpN manifold.

4. Golden deformation on para-Sasaki-like para-Norden manifolds

Definition 4.1. *An apcpN manifold* (M, ϕ, ξ, η, g) *is called a para-Sasaki-like para-Norden manifold if the structure tensors* (ϕ, ξ, η, g) *satisfy the following conditions for* $x, w, z \in H = \ker(\eta)$ *[10]:*

$$\begin{aligned} F(x, w, z) &= F(\xi, w, z) = F(\xi, \xi, z) = 0, \\ F(x, w, \xi) &= -g(x, w). \end{aligned} \tag{12}$$

The class of para-Sasaki-like para-Norden manifolds is defined and examined in [10]. This particular subclass of the examined manifolds is determined by:

$$\begin{aligned} (\nabla_x \phi) w &= -g(x, w)\xi - \eta(w)x + 2\eta(x)\eta(w)\xi \\ &= -g(\phi x, \phi w)\xi - \eta(w)\phi^2 x. \end{aligned} \tag{13}$$

In this section, we focus on the para-Sasaki-like para-Norden manifolds. These manifolds have also been called para-Sasaki-like Riemannian manifolds and para-Sasaki-like Riemannian $\Pi-$manifolds in [3, 10]. In [10], the following identities are proved:

$$\begin{array}{ll} \nabla_x \xi = \phi x, & (\nabla_x \eta) w = g(x, \phi w), \\ R(x, w)\xi = -\eta(w)x + \eta(x)w, & R(\xi, w)\xi = \phi^2 w, \\ Ric(x, \xi) = -2n\eta(x), & Ric(\xi, \xi) = -2n, \end{array} \tag{14}$$

where R and Ric denote the curvature tensor and the Ricci tensor, respectively.

The distribution $H = \ker(\eta)$ is a $2n-$dimensional paracontact distribution of a para-Sasaki-like para-Norden manifold and equipped with an almost paracomplex structure $P = \phi_{|H}$, a metric $h = g_{|H}$, are the restrictions of ϕ, g on paracontact distribution H, respectively [10]. The metric h is pure according to P as follow

$$h(Px, w) = h(x, Pw) \tag{15}$$

or equivalently

$$h(Px, Pw) = h(x, w). \tag{16}$$

Remember that an almost paracomplex manifold of dimension $2n$, equipped with a para-Norden metric h satisfying (15) is called as an almost para-Norden manifold [15]. On para-Sasaki-like para-Norden manifold, the equality (13) is written in form $F(x, y, z)$ as

$$\begin{aligned} F(x, w, z) &= g((\nabla_x \phi) w, z) \\ &= -g(x, w)\eta(z) - \eta(w)g(x, z) + 2\eta(x)\eta(w)\eta(z). \end{aligned}$$

F becomes zero on paracontact distribution $H = \ker(\eta)$ of a para-Sasaki-like para-Norden manifold. So, $\nabla_x \phi = 0$ for every $x, w, z \in H$. In [13–15], the authors show that an almost para-Norden manifold is paraholomorphic Norden manifold if and only if the almost paracomplex structure is parallel with respect to the Riemannian connection. From here we see that the paracontact distribution of a para-Sasaki-like para-Norden manifold induces a $2n-$dimensional paraholomorphic Norden manifold.

On almost paracontact para-Norden manifold, the Nijenhuis tensor of the structure (ϕ, ξ, η) is determined by:

$$\begin{aligned} N(x, w) &= [\phi, \phi](x, w) - d\eta(x, w)\xi \\ &= [\phi x, \phi w] + \phi^2[x, w] - \phi[\phi x, w] - \phi[x, \phi w] - ((\nabla_x \eta)w)\xi + ((\nabla_w \eta)x)\xi. \end{aligned}$$

An almost paracontact structure (ϕ, ξ, η) is normal if and only if its Nijenhuis tensor vanishes [2, 18]. The para-Sasaki-like para-Norden manifold (M, ϕ, ξ, η, g) is a normal paracontact para-Norden manifold,

$N = 0$ [10]. The Nijenhuis tensor of the structure $(\Phi,\ \xi,\ \eta)$ on almost paracontact para-Norden manifold is obtained as

$$\begin{aligned} N_\Phi = &\frac{5}{4}N_\phi + \frac{5}{4}((\nabla_{\phi x}\eta)w)\xi + \frac{5}{4}\eta(w)\nabla_{\phi x}\xi - \frac{5}{4}\eta(w)(\nabla_\xi\phi)x \\ &- \frac{5}{4}((\nabla_{\phi w}\eta)x)\xi - \frac{5}{4}\eta(x)\nabla_{\phi w}\xi + \frac{5}{4}\eta(x)(\nabla_\xi\phi)w \\ &+ \frac{5}{4}\eta(x)((\nabla_\xi\eta)w)\xi - \frac{5}{4}\eta(w)((\nabla_\xi\eta)x)\xi + \frac{5}{4}\eta(x)\phi(\nabla_w\xi) \\ &+ \frac{5}{4}\eta((\nabla_w\phi)x)\xi + \frac{5}{4}\eta(x)\eta(\nabla_w\xi)\xi - \frac{5}{4}\eta(w)\phi(\nabla_x\xi) \\ &- \frac{5}{4}\eta((\nabla_x\phi)w)\xi - \frac{5}{4}\eta(w)\eta(\nabla_x\xi)\xi - ((\nabla_x\eta)w)\xi + ((\nabla_w\eta)x)\xi. \end{aligned}$$

Considering (1), (2), (13) and (14),

$$N_\Phi = \frac{5}{4}N_\phi$$

is obtained on para-Sasaki-like para-Norden manifold. Then, the following theorem is written.

Theorem 4.2. *The structure (Φ, ξ, η) is a normal on para-Sasaki-like para-Norden manifold.*

Theorem 4.3. *If $\left(M, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ is a Golden-deformed on para-Sasaki-like para-Norden manifold, then the relation between the Riemannian connections $\widetilde{\nabla}$ of the metrics $\widetilde{g}$ and ∇ of the metric g is given by*

$$\widetilde{\nabla}_x w = \nabla_x w + \frac{5-3\sqrt{5}}{4} g\left(\phi x, \phi w\right)\xi - \frac{5-3\sqrt{5}}{4} g\left(\phi x, w\right)\xi. \tag{17}$$

Proof. Utilizing the general Kozsul formula, the equation

$$\begin{aligned} 2\widetilde{g}\left(\widetilde{\nabla}_x w, z\right) &= x\widetilde{g}(w,z) + w\widetilde{g}(z,x) - z\widetilde{g}(x,w) \\ &+\widetilde{g}([x,w],z) + \widetilde{g}([z,x],w) + \widetilde{g}([z,w],x) \end{aligned}$$

is written on a Golden-deformed para-Sasaki-like para-Norden manifold. Considering (2), (4), (11) and (14), the relation between $\widetilde{\nabla}$ and ∇ is reached. □

Proposition 4.4. *Let $\left(M, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ be a Golden-deformed para-Sasaki-like para-Norden manifold. The following equalities hold:*

$$\begin{aligned} &\widetilde{\nabla}_x\widetilde{\xi} = -\tfrac{1-\sqrt{5}}{2}\phi x, \\ &\left(\widetilde{\nabla}_x\widetilde{\eta}\right)w = -\tfrac{3-3\sqrt{5}}{4} g\left(\phi x, w\right) + \tfrac{5-\sqrt{5}}{4} g\left(\phi x, \phi w\right), \\ &\left(\widetilde{\nabla}_x\widetilde{\phi}\right)w = \left(\nabla_x\phi\right)w + \tfrac{5-3\sqrt{5}}{4} g\left(\phi x, w\right)\xi - \tfrac{5-3\sqrt{5}}{4} g\left(\phi x, \phi w\right)\xi. \end{aligned} \tag{18}$$

Proof. Using (11) *and* (14), *the assertions in* (18) *is directly obtain.* □

Taking into account (17), $(0,3)$-tensor field $\widetilde{F}$ has the following form:

$$\widetilde{F}(x,y,z) = \frac{3}{2}F(x,w,z) - \frac{\sqrt{5}}{2} g\left(\phi x, z\right)\eta(w) - \frac{\sqrt{5}}{2} g\left(\phi x, w\right)\eta(z). \tag{19}$$

It is seen that the Riemannian connections $\widetilde{\nabla}$ and ∇ coincide on paraholomorphic Norden manifold for $x, w \in H$. Moreover, $\widetilde{F}(x,w,z) = \frac{3}{2}F(x,w,z)$ holds on paraholomorphic Norden manifold. Considering the Definition 4.1 and equation (19), a Golden-deformed para-Sasaki-like para-Norden manifold is a para-Sasaki-like para-Norden manifold.

On a Golden-deformed para-Sasaki-like para-Norden manifold, the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ is defined as follows:

$$\widetilde{R}(x,w)z = \widetilde{\nabla}_x\widetilde{\nabla}_w z - \widetilde{\nabla}_w\widetilde{\nabla}_x z - \widetilde{\nabla}_{[x,w]}z.$$

Taking into account (14) and (17), the following relation between the corresponding curvature tensors $\widetilde{R}$ and R of the Riemannian connections $\widetilde{\nabla}$ and ∇, respectively, is obtained:

$$\begin{aligned}\widetilde{R}(x,w)z &= R(x,w)z \\ &+\frac{5-3\sqrt{5}}{4}\left(g\left(\phi x,\phi z\right)\eta(w)\xi + g\left(w,\phi z\right)\eta(x)\xi\right) \\ &+\frac{5-3\sqrt{5}}{4}\left(g\left(\phi w,\phi z\right)\phi x + g\left(\phi x,z\right)\phi w\right) \\ &-\frac{5-3\sqrt{5}}{4}\left(g\left(\phi w,\phi z\right)\eta(x)\xi + g\left(x,\phi z\right)\eta(w)\xi\right) \\ &-\frac{5-3\sqrt{5}}{4}\left(g\left(\phi x,\phi z\right)\phi w + g\left(\phi w,z\right)\phi x\right)\end{aligned} \tag{20}$$

On apcpN manifold, the Ricci tensor Ric, the scalar curvature $Scal$ and the $*-$scalar curvature $Scal^*$ are defined as usual traces of the $(0,4)-$type curvature tensor $R(x,w,z,y) = g(R(x,w)z,y)$,

$$Ric(x,w) = \sum_{i=0}^{2n} R(e_i,x,w,e_i),$$
$$Scal = \sum_{i=0}^{2n} Ric(e_i,e_i),$$
$$Scal^* = \sum_{i=0}^{2n} Ric(e_i,\varphi e_i),$$

with respect to an arbitrary ortonormal basis $\{e_0,...,e_{2n}\}$ of its tangent space [10]. On account of (20), the Ricci tensor $\widetilde{Ric}$, the scalar curvature tensor $\widetilde{Scal}$ and $*-$scalar curvature tensor $\widetilde{Scal}^*$ are obtained on Golden-deformed para-Sasaki-like para-Norden manifold as

$$\begin{aligned}\widetilde{Ric}(x,w) &= Ric(x,w), \\ \widetilde{Scal} &= \tfrac{3}{2}Scal - \tfrac{\sqrt{5}}{2}Scal^* + 2n\tfrac{\sqrt{5}}{2}, \\ \widetilde{Scal}^* &= \tfrac{3}{2}Scal^* - \tfrac{\sqrt{5}}{2}Scal - 2n\tfrac{\sqrt{5}}{2}\ .\end{aligned} \tag{21}$$

5. An example

Consider a $5-$dimensional real connected Lie group denoted by L. Then, the Lie group L has a basis of left-invariant vector fields $\{e_0,\ e_1,\ e_2,\ e_3,\ e_4\}$ with associated Lie algebra determined as follows:

$$[e_0,\ e_1] = -e_3,\quad [e_0,\ e_2] = -e_4,\quad [e_0,\ e_3] = -e_1,\quad [e_0,\ e_4] = -e_2. \tag{22}$$

The Lie group L is endowed with an almost paracontact para-Norden structure $\left(\phi,\xi,\eta,g\right)$ as follows:

$$\begin{aligned}\xi &= e_0,\quad \phi e_0 = 0,\quad \phi e_1 = e_3,\quad \phi e_2 = e_4, \\ \phi e_3 &= e_1,\quad \phi e_4 = e_2, \\ g(e_0,e_0) &= g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = 1, \\ g\left(e_i,e_j\right) &= 0,\ i,j\in\{0,1,2,3,4\},\ i\neq j.\end{aligned} \tag{23}$$

In [10], it is showed that the solvable Lie group corresponding to the Lie algebra defined by (22) and equipped with the almost paracontact para-Norden structure (ϕ,ξ,η,g) from (23) is a para-Sasaki-like para-Norden manifold. Morever, the basic components of ∇ and R are obtained. The components of the Riemannian connection ∇ are determined via Kozsul formula as follows:

$$\begin{array}{lllll}
\nabla_{e_0}e_0=0, & \nabla_{e_0}e_1=0, & \nabla_{e_0}e_2=0, & \nabla_{e_0}e_3=0, & \nabla_{e_0}e_4=0,\\
\nabla_{e_1}e_0=e_3, & \nabla_{e_1}e_1=0, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_3=-e_0, & \nabla_{e_1}e_4=0,\\
\nabla_{e_2}e_0=e_4, & \nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=0, & \nabla_{e_2}e_3=0, & \nabla_{e_2}e_4=-e_0,\\
\nabla_{e_3}e_0=e_1, & \nabla_{e_3}e_1=-e_0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_3=0, & \nabla_{e_3}e_4=0,\\
\nabla_{e_4}e_0=e_2, & \nabla_{e_4}e_1=0, & \nabla_{e_4}e_2=-e_0, & \nabla_{e_4}e_3=0, & \nabla_{e_4}e_4=0.
\end{array}$$

On a para-Sasaki-like para-Norden manifold, the non-zero components of the fundamental tensor F are obtained in the following way:

$$F_{110}=F_{101}=F_{220}=F_{202}=F_{330}=F_{303}=F_{440}=F_{404}=-1.$$

On a para-Sasaki-like para-Norden manifold, the non-zero components of the curvature tensor R corresponding to the Riemannian connection ∇ are expressed by:

$$\begin{array}{lll}
R(e_0,e_4)e_0 & = & R(e_1,e_2)e_3=-R(e_2,e_4)e_2=-R(e_2,e_3)e_1=e_4,\\
R(e_0,e_4)e_4 & = & R(e_0,e_2)e_2=R(e_0,e_1)e_1=R(e_0,e_3)e_3=-e_0,\\
R(e_0,e_3)e_0 & = & -R(e_1,e_2)e_4=-R(e_1,e_3)e_1=-R(e_1,e_4)e_2=e_3,\\
R(e_0,e_1)e_0 & = & R(e_1,e_3)e_3=R(e_2,e_3)e_4=-R(e_3,e_4)e_2=e_1,\\
R(e_0,e_2)e_0 & = & R(e_1,e_4)e_3=R(e_2,e_4)e_4=R(e_3,e_4)e_1=e_2,
\end{array}$$

On a Golden-deformed para-Sasaki-like para-Norden manifold, the non-zero components of the Riemannian connection $\widetilde{\nabla}$ is given by

$$\begin{array}{ll}
\widetilde{\nabla}_{e_1}e_0=e_3, & \widetilde{\nabla}_{e_2}e_0=e_4,\\
\widetilde{\nabla}_{e_1}e_1=\widetilde{\nabla}_{e_2}e_2=\widetilde{\nabla}_{e_3}e_3=\widetilde{\nabla}_{e_4}e_4=\frac{5-3\sqrt{5}}{4}e_0, & \widetilde{\nabla}_{e_3}e_0=e_1,\\
\widetilde{\nabla}_{e_1}e_3=\widetilde{\nabla}_{e_2}e_4=\widetilde{\nabla}_{e_3}e_1=\widetilde{\nabla}_{e_4}e_2=-\frac{9-3\sqrt{5}}{4}e_0, & \widetilde{\nabla}_{e_4}e_0=e_2.
\end{array}$$

On a Golden-deformed para-Sasaki-like para-Norden manifold, the non-zero components of the fundamental tensor $\widetilde{F}$ are obtained in the following way:

$$\begin{array}{l}
\widetilde{F}_{101}=\widetilde{F}_{110}=\widetilde{F}_{202}=\widetilde{F}_{220}=\widetilde{F}_{330}=\widetilde{F}_{303}=\widetilde{F}_{440}=\widetilde{F}_{404}=-\frac{3}{2},\\
\widetilde{F}_{103}=\widetilde{F}_{240}=\widetilde{F}_{204}=\widetilde{F}_{310}=\widetilde{F}_{301}=\widetilde{F}_{420}=\widetilde{F}_{402}=\widetilde{F}_{130}=-\frac{\sqrt{5}}{2}.
\end{array}$$

The manifold (L,ϕ,ξ,η,g) is the para-Sasaki-like para-Norden manifold since it satisfies the conditions (12). Hence, the manifold $(L,\widetilde{\phi},\widetilde{\xi},\widetilde{\eta},\widetilde{g})$ is also the para-Sasaki-like para-Norden manifold. If $x=a_0e_0+a_1e_1+a_2e_2+a_3e_3+a_4e_4$, $w=b_0e_0+b_1e_1+b_2e_2+b_3e_3+b_4e_4$, then the metric $\widetilde{g}$ is given by

$$\widetilde{g}(x,w)=\frac{3}{2}(a_0b_0+a_1b_1+a_2b_2+a_3b_3+a_4b_4)+\frac{\sqrt{5}}{2}(a_0b_0+a_1b_3+a_2b_4+a_3b_1+a_4b_2).$$

On a Golden-deformed para-Sasaki-like para-Norden manifold, the non-zero components of the curvature tensor $\widetilde{R}$ corresponding to the Riemannian connection $\widetilde{\nabla}$ are expressed by:

$$\begin{aligned}
&\widetilde{R}(e_0,e_1)e_0 = e_1,\\
&\widetilde{R}(e_0,e_2)e_0 = e_2,\\
&\widetilde{R}(e_0,e_3)e_0 = e_3,\\
&\widetilde{R}(e_0,e_4)e_0 = e_4,\\
&\widetilde{R}(e_0,e_1)e_1 = \widetilde{R}(e_0,e_4)e_4 = \widetilde{R}(e_0,e_2)e_2 = \widetilde{R}(e_0,e_3)e_3 = -\tfrac{9-3\sqrt{5}}{4}e_0,\\
&\widetilde{R}(e_1,e_4)e_3 = \widetilde{R}(e_3,e_4)e_1 = \tfrac{9-3\sqrt{5}}{4}e_2,\\
&\widetilde{R}(e_2,e_4)e_2 = -\tfrac{5-3\sqrt{5}}{4}e_2 - \tfrac{9-3\sqrt{5}}{4}e_4,\\
&\widetilde{R}(e_2,e_3)e_3 = -\widetilde{R}(e_1,e_2)e_1 = \tfrac{5-3\sqrt{5}}{4}e_4,\\
&\widetilde{R}(e_3,e_4)e_4 = -\widetilde{R}(e_2,e_3)e_2 = \tfrac{5-3\sqrt{5}}{4}e_1,\\
&\widetilde{R}(e_3,e_4)e_2 = -\widetilde{R}(e_2,e_3)e_4 = -\tfrac{9-3\sqrt{5}}{4}e_1,\\
&\widetilde{R}(e_1,e_2)e_2 = \widetilde{R}(e_1,e_4)e_4 = \tfrac{5-3\sqrt{5}}{4}e_3,\\
&\widetilde{R}(e_1,e_2)e_3 = -\widetilde{R}(e_2,e_3)e_1 = \tfrac{9-3\sqrt{5}}{4}e_4,\\
&\widetilde{R}(e_0,e_3)e_1 = \widetilde{R}(e_0,e_2)e_4 = \widetilde{R}(e_0,e_4)e_2 = \widetilde{R}(e_0,e_1)e_3 = \tfrac{5-3\sqrt{5}}{4}e_0,\\
&\widetilde{R}(e_1,e_3)e_3 = \tfrac{9-3\sqrt{5}}{4}e_1 + \tfrac{5-3\sqrt{5}}{4}e_3,\\
&\widetilde{R}(e_1,e_3)e_1 = -\tfrac{5-3\sqrt{5}}{4}e_1 - \tfrac{9-3\sqrt{5}}{4}e_3,\\
&\widetilde{R}(e_1,e_4)e_1 = -\widetilde{R}(e_4,e_3)e_3 = -\tfrac{5-3\sqrt{5}}{4}e_2,\\
&\widetilde{R}(e_1,e_4)e_2 = \widetilde{R}(e_1,e_2)e_4 = -\tfrac{9-3\sqrt{5}}{4}e_3,\\
&\widetilde{R}(e_2,e_4)e_4 = \tfrac{9-3\sqrt{5}}{4}e_2 + \tfrac{5-3\sqrt{5}}{4}e_4.
\end{aligned}$$

6. Conclusions

In this paper, we define Golden deformation on almost paracontact para-Norden manifolds by establishing the relationship between Golden structure and paracontact structure. We obtain the Riemannian connection on Golden-deformed para-Sasaki-like para-Norden manifolds and compute the curvature tensor, scalar curvature and $*$-scalar curvature based on this Riemannian connection. Finally, we give an example based on the results of the paper.

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