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Jaggi type contraction mappings in interpolative spaces

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Abstract. In the current paper we provide a more general fixed point result under a relaxed contractive condition inspired by Jaggi-type contractions within the framework of interpolative metric spaces. For these mappings we prove that they possess a fixed point under certain conditions. Also, we give a condition for the uniqueness of the fixed point. Moreover, as consequences, we provide two fixed point results in interpolative metric spaces, where the fixed point is unique.

1. Introduction and Preliminaries

The study of fixed point results under more relaxed contractive conditions and in more general metric spaces has been of great interest for the research community in the past years (see, for example [12] and the references cited herein). Motivated by the recently introduced types of metric space introduced by Karapınar in [13], namely interpolative metric spaces, and the study of Jaggi type contractions in different contexts (see [6], [1],[5], [9], [11], [17]), in this paper we aim to study Jaggi type contractions in the sense of [8] in interpolative metric spaces.

Very recently, Karapınar introduced in [14] the notion of interpolative metric spaces defined by a modified triangle inequality that includes parameters α and c. These new metric spaces, which emerged from the concept of interpolative contractions (see [13]) enable broader applications in fixed-point theory. It is proven in [14] that fixed points in such spaces exist under certain conditions. Let us recall the definition of an interpolative metric space (see [13, 14]):

Definition 1.1. *Let X be a nonempty set. We say that $d : X \times X \to [0, +\infty)$ is (α, c)-interpolative metric if*

(m_1) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,
(m_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(m_3) *there exist an* $\alpha \in (0, 1)$ *and* $c \geq 0$ *such that*

$$d(x, y) \leq d(x, z) + d(z, y) + c\,(d(x, z))^{\alpha}\,(d(z, y))^{1-\alpha},$$

for all $(x, y, z) \in X \times X \times X$.

Then, we call (X, d) *an* (α, c)*-interpolative metric space.*

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In these new metric spaces there has been proved an analogue of Banach's fixed point theorem as follows:

Theorem 1.2 (see [13]). *Let (X, d) be a (α, c)-interpolative metric space and let $T : X \to X$ be a mapping such that exists $q \in (0, 1)$ such that*

$$d(Tx, Ty) \leq qd(x, y)$$

for all $x, y \in X$. Then, T possesses a unique fixed point.

More results on interpolative metric spaces can be found in [15] and [4].

Remark 1.3. *Let us note that every metric space can be regarded as an (α, c)-interpolative metric space with $c = 0$. However, the reverse of this statement does not hold.*

Indeed, consider the metric as in Example 1.1 from [13] $d : X \times X \to [0, \infty)$ defined as

$$d(x, y) := \delta(x, y)(\delta(x, y) + 1).$$

where (X, δ) is a standard metric space.

Let us first introduce some preliminary notions related to interpolative metric spaces.

Definition 1.4. *(see [13]) Let (X, d) be a (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that*

i) $\{x_n\}$ *converges to x in X, if and only if, $d(x_n, x) \to 0$, as $n \to \infty$.*

ii) $\{x_n\}$ *is a Cauchy sequence in X, if and only if,* $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$.

iii) (X, d) *is a complete (α, c)-interpolative metric space if every Cauchy sequence converges in X.*

For completeness, we recall some notions related to admissible mappings:

Definition 1.5 (see [10]). *For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$.*

(i) *(see [19]) We say that T is α-admissible if*

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

(ii) *(see [7]) A self-mapping T is called triangular α-admissible if*

1. *T is α-admissible, and*
2. *$\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ imply $\alpha(x, z) \geq 1$, for any $x, y, z \in X$.*

(iii) *(see [2]) Let $S : X \to X$ be a mapping. We say that (T, S) is a generalized α-admissible pair if for all $x, y \in X$, we have*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Sy) \geq 1 \text{ and } \alpha(STx, TSy) \geq 1.$$

(iv) *(see [16]) We say that T is α-orbital admissible if*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Also, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;
(Ψ_2) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in [0, \infty)$.

Here, these functions are called as (c)-comparison (see e.g. [18]). In [18], Rus proved that if $\psi \in \Psi$, then the following hold:

(a) the sequence $(\psi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in [0, \infty)$;
(b) the strict inequality, $\psi(t) < t$, holds for any $t \in [0, \infty)$;
(c) the function ψ is continuous at 0;
(d) the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in [0, \infty)$.

Jaggi, in [6], introduced a fixed-point result that generalizes Banach's fixed-point theorem (see [3]) by employing a more general contractive condition.

Theorem 1.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous map such that there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that:*

$$d(Tt, Ts) \leq \alpha \frac{d(s, Ts)(1 + d(x, Tx))}{1 + d(t, s)} + \beta d(t, s)$$

holds for every distinct $t, s \in X$. Then T has a unique fixed point.

These mappings were of great interest to the research community, as many generalized results appeared based on Jaggi type mappings.

In [8], there were introduced a new type of mappings called Jaggi type based on rational contraction and admissible mappings, in the context of complete partial metric spaces, as follows:

Definition 1.7 (see [8]). *Let (M, p) be a complete partial metric space and $T : M \to M$ be a map. The map T will be referred to as a map of $(\alpha - \psi)$-Jaggi type if there exist a function $\psi \in \Psi$, and nonnegative real numbers a_1, a_2 with $a_1 + a_2 < 1$, so that the inequality*

$$\alpha(t, s)p(Tt, Ts) \leq \psi\left(\frac{a_1 p(t, Tt) \cdot p(s, Ts)}{p(t, s)} + a_2 p(t, s)\right)$$

holds for every distinct $t, s \in M$.

In this paper, we investigate the existence and uniqueness of fixed points of Jaggi type mappings in the setting of complete interpolative metric spaces.

2. Main Results

Let us begin with the definition of $(\alpha - \psi)$-Jaggi type contractions within interpolative metric spaces:

Definition 2.1. *Let (X, d) be an interpolative metric space and $T : X \to X$ be a map. Then T will be referred to as a map of $(\alpha - \psi)$-Jaggi type in interpolative metric spaces if there exist $\psi \in \Psi$, and nonnegative real numbers a_1, a_2 with $a_1 + a_2 < 1$ so that the inequality*

$$\alpha(t, s)d(Tt, Ts) \leq \psi\left(a_1 \frac{d(t, Tt) \cdot d(s, Ts)}{d(t, s)} + a_2 d(t, s)\right) \tag{1}$$

holds for every distinct $t, s \in M$.

Lemma 2.2 (see [8]). *Let X be a non-empty set. Suppose that $\alpha : X \times X \to [0, \infty)$ is a function and $T : X \to X$ is an α-orbital admissible mapping. If there exists $t_0 \in X$ such that $\alpha(t_0, Tt_0) \geq 1$, and $t_n = Tt_{n-1}$ for $n = 0, 1, \ldots$, then, we have*

$$\alpha(t_n, t_{n+1}) \geq 1, \text{ for each } n = 0, 1, \ldots. \tag{2}$$

Theorem 2.3. *Let (X, d) be a complete interpolative metric space, $T : X \to X$ be a map of $(\alpha - \psi)$-Jaggi type and there exists $t_0 \in X$ such that $\alpha(t_0, Tt_0) \geq 1$. If the α-orbital admissible mapping T is continuous, then T has a fixed point in X.*

Proof. Given the assumption of the theorem, there exists $t_0 \in X$ such that $\alpha(t_0, Tt_0) \geq 1$. Thus, we can construct a sequence as follows:

$$t_n = Tt_{n-1} \quad \text{for} \quad n = 0, 1, \ldots$$

According to Lemma 2.2, we obtain (2).

Without loss of generality, we may assume that

$$d(t_n, t_{n+1}) > 0 \quad \text{for each} \quad n = 0, 1, \ldots \tag{3}$$

Indeed, if there exists some k_0 such that $d(t_n, t_{n+1}) = 0$, then by (m_1) we have $t_{k_0} = t_{k_0+1} = Tt_{k_0}$. This implies that $u = t_{k_0}$ is a fixed point of T, completing the proof.

Thus, throughout the proof, we maintain (3). Consequently, we can apply the inequality (1) for any successive terms $t = t_n$ and $s = t_{n+1}$:

$$\begin{aligned} d(t_{n+1}, t_{n+2}) &= d(Tt_n, Tt_{n+1}) \leq \alpha(t_n, t_{n+1}) d(Tt_n, Tt_{n+1}) \\ &\leq \psi\left(a_1 \tfrac{d(t_n, Tt_n) \cdot d(t_{n+1}, Tt_{n+1})}{d(t_n, t_{n+1})} + a_2 d(t_n, t_{n+1})\right) \\ &= \psi\left(a_1 \tfrac{d(t_n, t_{n+1}) \cdot d(t_{n+1}, t_{n+2})}{d(t_n, t_{n+1})} + a_2 d(t_n, t_{n+1})\right) \\ &= \psi\left(a_1 d(t_{n+1}, t_{n+2}) + a_2 d(t_n, t_{n+1})\right) \end{aligned} \tag{4}$$

We need to consider two cases:

(i) $d(t_{n+1}, t_{n+2}) > d(t_n, t_{n+1})$

(ii) $d(t_{n+1}, t_{n+2}) \leq d(t_n, t_{n+1})$

If the first case occurs for some n, then due to condition (Ψ_1) and the fact that $\psi(t) < t$ for all $t > 0$, inequality (4) becomes:

$$\begin{aligned} d(t_{n+1}, t_{n+2}) &\leq \psi\left([a_1 + a_2] d(t_{n+1}, t_{n+2})\right) \\ &\leq \psi\left(d(t_{n+1}, t_{n+2})\right) < d(t_{n+1}, t_{n+2}) \end{aligned} \tag{5}$$

This leads to a contradiction. Hence, the second case $d(t_{n+1}, t_{n+2}) \leq d(t_n, t_{n+1})$ holds for all $n \in \mathbb{N}_0$.

Moreover, inequality (4) implies that:

$$d(t_{n+1}, t_{n+2}) \leq \psi\left([a_1 + a_2] d(t_n, t_{n+1})\right) \leq \psi\left(d(t_n, t_{n+1})\right) < d(t_n, t_{n+1}) \tag{6}$$

Thus, iteratively we obtain:

$$d(t_{n+1}, t_n) \leq \psi^n(d(t_1, t_0)) \quad \text{for all} \quad n \geq 1. \tag{7}$$

Since $\psi \in \Psi$ we get

$$\lim_{n \to \infty} d(t_{n+1}, t_n) = 0 \tag{8}$$

By (m_3) we have:

$$d(t_n, t_{n+2}) \leq d(t_n, t_{n+1}) + d(t_{n+1}, t_{n+2}) + c\left(d(t_n, t_{n+1})^{\alpha}(d(t_{n+1}, t_{n+2}))^{1-\alpha}\right),$$

and taking $n \to \infty$ and considering the limit (8), we find that

$$\lim_{n\to\infty} d(t_n, t_{n+2}) = 0.$$

Moreover, we have that

$$d(t_n, t_{n+3}) \leq d(t_n, t_{n+2}) + d(t_{n+2}, t_{n+3}) + c\left(d(t_n, t_{n+2})^{\alpha}(d(t_{n+2}, t_{n+3}))^{1-\alpha}\right) \tag{9}$$

By combining the limit (8) and inequality (9), we deduce that

$$\lim_{n\to\infty} d(t_n, t_{n+3}) = 0.$$

By induction, we obtain that

$$\lim_{n\to\infty} d(t_n, t_{n+r}) = 0. \tag{10}$$

$$d(t_n, t_{n+r+1}) \leq d(t_n, t_{n+r}) + d(t_{n+r}, t_{n+r+1}) + c\left(d(t_n, t_{n+r})^{\alpha}(d(t_{n+r}, t_{n+r+1}))^{1-\alpha}\right)$$

by employing the limits (8) and (10), as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(t_n, t_{n+r+1}) = 0.$$

Now, we have

$$d(t_{n+1}, t_m) < 1, \tag{11}$$

for $m > n > k$, and we shall prove that the constructed iterative sequence $\{t_n\}$ is a Cauchy sequence. For $m > n > k$, let us make the notation $\psi(d(t_k, t_{k+1})) = q$, then, we have

$$\begin{aligned} d(t_n, t_m) &\leq d(t_n, t_{n+1}) + d(t_{n+1}, t_m) + c\left[(d(t_n, t_{n+1}))^{\alpha}(d(t_{n+1}, t_m))^{1-\alpha}\right] \\ &\leq q^{n-k}d(t_k, t_{k+1}) + d(t_{n+1}, t_m) + c\left[q^{n-k}d(t_k, t_{k+1})^{\alpha}(d(t_{n+1}, t_m))^{1-\alpha}\right]. \end{aligned} \tag{12}$$

Since $d(t_{n+1}, t_m) < 1$ we have $(d(t_{n+1}, t_m))^{1-\alpha} < 1$. Thus, the right-hand side of the previous inequality can be estimated as follows:

$$\begin{aligned} &\leq q^{n-k}d(t_k, t_{k+1}) + \left[(d(t_{n+1}, t_m))^{\alpha} + c(q^{n-k})^{\alpha}\right] d(t_{n+1}, t_m)d(t_k, t_{k+1}) \\ &\leq q^{n-k} + \left[1 + c(q^{n-k})^{\alpha}\right] d(t_{n+1}, t_m). \end{aligned} \tag{13}$$

In conclusion, we have

$$d(t_n, t_m) \leq q^{n-k} + \left[1 + c(q^{n-k})^{\alpha}\right] d(t_{n+1}, t_m). \tag{14}$$

Notice also that

$$d(t_{n+1}, t_m) \le d(t_{n+1}, t_{n+2}) + d(t_{n+2}, t_m) + c\left[(q^{n-k+1} d(t_k, t_{k+1}))^{\alpha} (d(t_{n+2}, t_m))^{1-\alpha}\right]$$
$$\le q^{n-k+1} d(t_k, t_{k+1}) + \left[(d(t_{n+2}, t_m))^{\alpha} + c(q^{n-k+1})^{\alpha}\right] d(t_{n+2}, t_m) d(t_k, t_{k+1}) \tag{15}$$
$$\le q^{n-k+1} + \left[1 + c(q^{n-k+1})^{\alpha}\right] d(t_{n+2}, t_m),$$

Thus, we have

$$d(t_n, t_m) \le q^{n-k} + q^{n-k+1}\left[1 + c(q^{n-k})^{\alpha}\right] + \left[1 + c(q^{n-k})^{\alpha}\right]\left[1 + c(q^{n-k+1})^{\alpha}\right] d(t_{n+2}, t_m). \tag{16}$$

so we obtain

$$d(t_n, t_m) \le q^{n-k} \sum_{i=0}^{m-n-1} q^i \prod_{j=0}^{i-1} (1 + cq^{n-k+j})^{\alpha} \le q^{n-k} \sum_{i=0}^{m-n-1} q^i \prod_{j=0}^{i-1} (1 + cq^j)^{\alpha}. \tag{17}$$

Since $q < 1$, the right-hand side of the previous inequality is dominated by the sequence $\sum_{i=0}^{\infty} \prod_{j=0}^{i-1} (1 + cq^j)^{\alpha}$, which is convergent.

Thus, $\{t_n\}$ is a Cauchy sequence and by completeness of (X, d), there exists $u \in X$ such that:

$$\lim_{n\to\infty} d(t_n, u) = 0 = \lim_{n\to\infty} d(t_n, t_{n+k}) = d(u, u) \tag{18}$$

Given that T is continuous, we conclude from (18) that:

$$\lim_{n\to\infty} d(t_{n+1}, Tu) = \lim_{n\to\infty} d(Tt_n, Tu) = 0, \tag{19}$$

we find that u is a fixed point of T, i.e., $Tu = u$. □

Similarly to [8], the continuity condition can be relaxed by imposing another condition.

Definition 2.4. *Let $s \ge 1$. We say that an interpolative metric spaces (X, d) is regular if $\{t_n\}$ is a sequence in X such that $\alpha(t_n, t_{n+1}) \ge 1$ for each n and $t_n \to t \in X$ as $n \to \infty$, then there is a subsequence $\{t_{n(k)}\}$ of $\{t_n\}$ such that $\alpha(t_{n(k)}, t) \ge 1$ for each k.*

Theorem 2.5. *Let (X, d) be a regular complete interpolative metric space, and $T : X \to X$ be a map of $(\alpha - \psi)$-Jaggi type. Assume there exists $t_0 \in X$ such that $\alpha(t_0, Tt_0) \ge 1$. If T is an α-orbital admissible mapping, then T has a fixed point in X.*

Remark 2.6. *If for all $t \ne s \in X$, there exists $r \in X$ such that*

$$\alpha(t, r) \ge 1, \quad \alpha(s, r) \ge 1, \quad \text{and} \quad \alpha(r, Tr) \ge 1.$$

the fixed point of T in Theorem 2.3 and Theorem 2.5 is unique.

3. Consequences

Definition 3.1. *Let (X, d) be an interpolative metric space, and let $T : X \to X$ be a map and $\psi \in \Psi$. Then T will be referred to as a map of ψ-Jaggi type if there exist a_1 and a_2 in $[0, 1)$ with $a_1 + a_2 < 1$ so that the inequality*

$$d(Tt, Ts) \le \psi\left(a_1 \frac{d(t, Tt) + d(s, Ts)}{d(t, s)} + a_2 d(t, s)\right) \tag{14}$$

holds for every distinct $t, s \in X$.

Theorem 3.2. *Let (X, d) be a complete interpolative metric space and $T : M \to M$ be a map of ψ-Jaggi type. If T is continuous, then T has a unique fixed point in M.*

Proof. Taking $\alpha(t, s) = 1$ for all $t, s \in X$, the conclusion follows from Theorem 2.2 and Remark 2.6. □

Theorem 3.3. *Let (X, d) be a complete interpolative metric space and $T : X \to X$ be a map of Jaggi type, satisfying the following condition:*

$$d(Tt, Ts) \leq \nu_1 \frac{d(t, Tt) \cdot d(s, Ts)}{d(t, s)} + \nu_2 d(t, s) \tag{15}$$

for all distinct $t, s \in X$, and for some $\nu_1, \nu_2 \in [0, 1)$ with $\nu_1 + \nu_2 < 1$. Then T has a unique fixed point in X.

Proof. Taking $\psi(t) = kt$, $k \in [0, 1)$ in Theorem 3.2 with $\nu_i = ka_i$, $i = 1, 2$. □

References

[1] R. Agarwal, E. Karapınar, D. O'Regan, A. Roldán-López-de-Hierro. *Fixed point theory in metric type spaces*. Cham: Springer, (2015).

[2] H. Aydi, M. Jellali, E. Karapınar, *Common fixed points for generalized a-implicit contractions in partial metric spaces: Consequences and application*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **109** (2015), 367–384.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae **3** (1922), 133-181.

[4] S. Bilazeroglu, C. Yalcin, *A fixed point theorem for Ćirić type contraction on interpolative metric spaces*, Journal of Nonlinear and Convex Analysis 25(7) (2024), 1717-1723.

[5] C. Chen, G. Joonaghany, Karapınar, E., Khojasteh, F, *On bilateral contractions*, Mathematics, **7**(6) (2019), 538.

[6] D. S. Jaggi, *Some unique fixed point theorems*, Indian Journal of Pure and Applied Mathematics **2**8 (1977), 223-–230.

[7] E. Karapınar, P. Kumam, P. Salimi, *On a α-ψ-Meir–Keeler contractive mappings*, Fixed Point Theory Appl. **94** (2013), 94.

[8] E. Karapınar, *On Jaggi type contraction mappings*, U.P.B. Sci. Bull., Series A **80** (2018), 49–62.

[9] E. Karapınar, A. Fulga, *A hybrid contraction that involves Jaggi type*, Symmetry, **11**(5) (2019), 715.

[10] E. Karapınar, R. Agarwal, S. Yeşilkaya, C. Wang, *Fixed-point results for Meir–Keeler type contractions in partial metric spaces: A survey*, Mathematics **10**(17) (2022), 3109.

[11] E. Karapınar, A. Fulga, *Discussion on the hybrid Jaggi-Meir-Keeler type contractions*, AIMS Mathematics **7**(7) (2022), 12702–12717.

[12] E. Karapınar, R.P. Agarwal, *Fixed Point Theory in Generalized Metric Spaces*, 13 (2023), Synthesis Lectures on Mathematics & Statistics, Springer Cham, 13.

[13] E. Karapınar, *An open discussion: Interpolative Metric Spaces*, Adv. Theory Nonlinear Anal. Appl. **7** (2023).

[14] E. Karapınar, *On interpolative metric spaces*, *Filomat* (2024).

[15] E. Karapınar, R.P. Agarwal, *Some fixed point results on interpolative metric spaces*, Nonlinear Anal. Real World Appl. **82** (2025), 104244.

[16] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl. (2014), 2014–190

[17] S. Park, *Improving many metric fixed point theorems*, Letters in Nonlinear Analysis and its Applications **2**(2) (2024), 35–61

[18] I.A.Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

[19] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. Theory Methods Appl. **75** (2012), 2154–2165.