Filomat 39:9 (2025), 3051–3067
https://doi.org/10.2298/FIL2509051B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The generalized Laplace transform method for a ψ-Caputo coupled system of Volterra integro-differential equations

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Abstract. In this paper, we deal with the coupled system of Volterra integro-differential equations of order (p, q), The novelty of the considered problem is that it has been investigated under the ψ-Caputo fractional derivatives, which is more general than the works based on the well-known fractional derivatives such as (Caputo fractional derivative, Caputo–Hadamard fractional derivative and Caputo–Katugampola fractional derivative) for different values of the function ψ. We use the generalized Laplace transform method to find the solution then we obtain results on uniqueness using Banach's fixed point theorem. Next, we examine different types of stabilities in the sense of Ulam-Hyers (UH) of the given problems. Finally, a concrete application is given to illustrate the effectiveness of our main results.

1. Introduction

Differential equations represent a significant area of mathematics with extensive applications in science. They are utilized in mathematical modeling, aiding in solving physical and engineering problems involving functions of one or more variables [1, 10, 15, 40]. These problems can include the propagation of heat or sound, fluid flow, elasticity, electrostatics, and electrodynamics, among others [18, 39]. For decades, research into methods for solving differential equations has been a key focus for scholars due to their critical applications across various scientific fields. The technique of using integral transforms has demonstrated its efficiency and applicability in solving ordinary and partial differential equations.

Fractional semilinear equations are crucial for modeling inhibitory and excitatory neuron activity, capturing complex dynamics and memory effects that traditional models miss. They enhance our understanding of neural processing and improve treatments for neurological disorders. In electric circuit analysis, these equations accurately represent complex systems by incorporating memory properties, leading to better insights into capacitor behavior and signal propagation, which aids in designing and optimizing electronic devices. [21, 25–29].

The Laplace transform method simplifies solving differential equations by converting them into algebraic equations, making them easier to handle. Classical derivatives can be represented as convolutions of

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2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A08, 34A12.

Keywords. ψ-Caputo fractional derivative, Volterra integral operator, Generalized Laplace transform method, Banach's fixed point theorem, Hyers–Ulam stability.

Received: 13 January 2025; Accepted: 14 January 2025

Communicated by Maria Alessandra Ragusa

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power functions for fractional differential equations, which helps facilitate Laplace transforms. However, this method has limitations in the variability of orders due to challenges in finding inverse transforms, especially for higher orders.

This transform is effective for various equations, including integro-differential, integral, fractional, and delay differential equations. It can also address initial and boundary value problems associated with differential equations and solve linear Caputo fractional-integro differential equations with multiple time delays. When using this technique, it is essential to understand several key properties of the Laplace transform for effective application. [3, 5, 9, 11, 13, 23]

Schaefer's fixed point theorem is an important tool in fractional differential equations (FDEs), particularly those using Caputo derivatives. It establishes the existence and uniqueness of solutions to nonlinear FDEs, often challenging to solve analytically. Specifically, the theorem helps demonstrate that, under certain conditions, a unique solution for Caputo fractional differential equations exists. By proving that a related function has a fixed point unchanged by the function can conclude the existence of a unique solution. [12, 36].

The concept of stability has drawn considerable attention across various research fields and applications, given its vital role in ensuring consistent and reliable outcomes. Stability is a system's ability to return to equilibrium after a disturbance or maintain consistent behavior over time.

In 1940, Stanislaw Ulam introduced a form of stability known as Ulam stability, which pertains to functional equations. Ulam posed a fundamental question: if a function approximately satisfies a given equation, can we identify a true solution that closely aligns with the approximate one This inquiry established the foundation for what is now known as Hyers-Ulam stability. [14, 19, 20, 33]. Many authors have recently investigated Ulam–Hyers stability and generalized stability in various research articles, such as [2, 30, 31]. These studies have used the Ulam–Hyers stability criteria to examine the stability of solutions in different fractional differential equations, emphasizing its significance in contemporary mathematical research.

In [16], Choukri Derbazi et al. investigated various qualitative properties of solutions, such as estimating the solutions, the continuous dependence of the solutions on initial conditions, and the existence and uniqueness of extremal solutions for the following problem:

$$\begin{cases} {}^{C}\mathbb{D}^{\mu,\Phi}_{a^+} z(l) + \omega {}^{C}\mathbb{D}^{\kappa,\Phi}_{a^+} z(l) = \mathbb{F}(l, z(l)), \quad l \in \Delta = [a, b] \\ z(a) = z_a, \end{cases}$$

where ${}^{C}\mathbb{D}^{\mu,\Phi}_{a^+}$ and ${}^{C}\mathbb{D}^{\kappa,\Phi}_{a^+}$ denote the Φ-Caputo fractional derivatives, with the orders μ and κ respectively such that $0 < \kappa < \mu \leq 1$, $\omega > 0$, $z_a \in \mathbb{R}$ and $\mathbb{F} \in C(\Delta \times \mathbb{R}, \mathbb{R})$.

Also in [17] Choukri Derbazi et al. proved the uniqueness and the $\mathbb{E}_\mu$-UH stability of solutions for the following Φ–Caputo fractional multi terms differential equation (Φ–Caputo FMTDE) of the form

$$\begin{cases} {}^{C}\mathbb{D}^{\mu,\Phi}_{u^+} m(l) + \varrho {}^{C}\mathbb{D}^{\kappa,\Phi}_{u^+} m(l) = \mathbb{H}(l, m(l)), \quad l \in \Sigma = [u, v] \\ m(u) = \theta, \end{cases}$$

where ${}^{C}\mathbb{D}^{\mu,\Phi}_{u^+}$ and ${}^{C}\mathbb{D}^{\kappa,\Phi}_{u^+}$ denote the Φ-Caputo fractional derivatives, with the orders μ and κ respectively such that $0 < \kappa < \mu \leq 1$, $\varrho > 0$, $\mathbb{H} \in C(\Sigma \times \mathbb{R}, \mathbb{R})$ and $\theta \in \mathbb{R}$. Φ–Caputo fractional multi terms differential system (Φ–CaputoFMTDS) of the type

$$\begin{cases} {}^{C}\mathbb{D}^{\mu_1,\Phi}_{u^+} m_1(l) + \varrho_1 {}^{C}\mathbb{D}^{\kappa_1,\Phi}_{u^+} m_1(l) = \mathbb{K}_1(l, m_1(l), m_2(l)), \quad l \in \Sigma = [u, v] \\ {}^{C}\mathbb{D}^{\mu_2,\Phi}_{u^+} m_2(l) + \varrho_2 {}^{C}\mathbb{D}^{\kappa_2,\Phi}_{u^+} m_2(l) = \mathbb{K}_2(l, m_1(l), m_2(l)), \\ m_1(u) = \theta_1, \quad m_2(u) = \theta_2. \end{cases}$$

such that $0 < \kappa_i < \mu_i \leq 1$, $\varrho_i > 0$, $\mathbb{K}_i \in C(\Sigma \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\theta_i \in \mathbb{R}$, $i = 1, 2$.

Inspired by the previously described work, we aim to develop the generalized Laplace transform method for a coupled system of Volterra integro-differential equations. Specifically, we will analyze the following coupled system along with its initial conditions:

$$\begin{cases} {}^{C}\mathfrak{D}^{p_1,\psi}\vartheta(\iota) = \xi_1 {}^{C}\mathfrak{D}^{q_1,\psi}\vartheta(\iota) + \hbar_1\left(\iota, \vartheta(\iota), \int_0^{\iota} g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma\right), \iota \in \Lambda := [0, T], \\ {}^{C}\mathfrak{D}^{p_2,\psi}\omega(\iota) = \xi_2 {}^{C}\mathfrak{D}^{q_2,\psi}\omega(\iota) + \hbar_2\left(\iota, \vartheta(\iota), \int_0^{\iota} g_2(\iota, \varsigma, \omega(\varsigma))d\varsigma\right), \iota \in \Lambda := [0, T], \\ {}^{C}\mathfrak{D}^{k,\psi}\vartheta(0) = \vartheta_0 \quad k = 0, ..., \alpha - 1, \\ {}^{C}\mathfrak{D}^{k,\psi}\omega(0) = \omega_0 \quad k = 0, ..., \beta - 1, \end{cases} \tag{1}$$

where $\xi_1, \xi_2 \in \mathbb{R}$, $\alpha - 1 < p_1, p_2 \leq \alpha$, $\beta - 1 < q_1, q_2 \leq \beta$, $\alpha, \beta \in \mathbb{Z}^+$, $\beta \leq \alpha$. The fractional derivative $\mathfrak{D}^{q,\psi}$ is the ψ-Caputo fractional derivative of order $q \in (0, 1)$, $\vartheta_0, \omega_0 \in \mathbb{R}$, and $\hbar : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \Lambda \times \mathbb{R} \to \mathbb{R}$ are two given functions.

The Laplace transform is beneficial for solving integro-differential equations involving integral and differential operators. This method simplifies these equations into algebraic equations, making them easier to solve. Once the solution in the Laplace domain is determined, the inverse Laplace transform retrieves the solution in the time domain. [32, 35].

The rest of this paper is structured in the following way. Section 2 is dedicated to the primary definitions and notations, we also review basic definitions, lemmas, and important properties of the generalized Laplace transformation. In Section 3 by the implementation of Banach's fixed point theorem, we study the uniqueness and different kinds of UH stability of the proposed problem. Finally, we present an example to demonstrate our main results in section 4.

2. Preliminaries

This section will introduce essential concepts and definitions that form the foundation for the main results discussed in the following sections.

We begin by defining $\Lambda = [0, T]$ and let $\mathcal{X} = C(\Lambda, \mathbb{R})$, which is a Banach space equipped with the norm $\|\vartheta\| = \sup\{|\vartheta(\iota)| : \iota \in \Lambda\}$. Next, we consider the product space $\mathcal{C} = \mathcal{X} \times \mathcal{X}$, which is also a Banach space. It is endowed with the norm $\|(\vartheta, \omega)\|_{\mathcal{C}} = \|\vartheta\| + \|\omega\|$, and an alternative expression for the norm is given by $\|(\vartheta, \omega)\| = \max\{\|\vartheta\|, \|\omega\|\}$.

Definition 2.1. *(The ψ-Riemann–Liouville fractional integral [24, 34]). Let $q > 0$, and let $\vartheta : \Lambda \to \mathbb{R}$ be an integrable function. Let $\psi \in C^1(\Lambda, \mathbb{R})$ be a continuous and differentiable increasing function such that $\psi'(\iota) \neq 0$ for all $\iota \in \Lambda$. The ψ-Riemann–Liouville fractional integral of ϑ of order q is defined as follows:*

$$\mathfrak{I}_{0^+}^{q,\psi}\vartheta(\iota) = \frac{1}{\Gamma(q)} \int_0^{\iota} (\psi(\iota) - \psi(\kappa))^{q-1} \psi'(\kappa)\vartheta(\kappa)d\kappa. \tag{2}$$

where $\Gamma(q)$ is the Gamma function.

Note that for $\psi(\iota) = \iota$ and $\psi(\iota) = \ln(\iota)$, Equation (2) is reduced to the Riemann–Liouville and Hadamard fractional integrals, respectively.

Definition 2.2. *(The ψ-Riemann–Liouville fractional derivative [24, 34]).Let $n \in \mathbb{N}$, and let ψ and ϑ be two functions that belong to $C^n(\Lambda, \mathbb{R})$. We assume that ψ is an increasing function and that $\psi'(\iota) \neq 0$ for all $\iota \in \Lambda$. The*

ψ-Riemann–Liouville fractional derivative of ϑ of order q is defined as follows:

$$
\begin{aligned}
{}^{RL}\mathfrak{D}_{0^+}^{q,\psi}\vartheta(\iota) &= \left(\frac{1}{\psi'(\iota)}\frac{d}{dt}\right)^n \mathfrak{I}_{0^+}^{n-q,\psi}\vartheta(\iota) \\
&= \frac{1}{\Gamma(n-q)}\left(\frac{1}{\psi'(\iota)}\frac{d}{dt}\right)^n \int_0^\iota (\psi(\iota)-\psi(\kappa))^{n-q-1}\psi'(\kappa)\vartheta(\kappa)d\kappa
\end{aligned}
$$

Definition 2.3. *[8] Let $q > 0$, $\vartheta \in C^{n-1}(\Lambda, \mathbb{R})$ and $\psi \in C^n(\Lambda, \mathbb{R})$ such that $\psi'(\iota) > 0$ for all $\iota \in \Lambda$. The ψ-Caputo fractional derivative of order q of the function ϑ is given by*

$$
{}^{C}\mathfrak{D}_{0^+}^{q,\psi}\vartheta(\iota) = \frac{1}{\Gamma(n-q)}\int_0^\iota \psi'(\kappa)(\psi(\iota)-\psi(\kappa))^{n-q-1}\vartheta_{[n]}^{\psi}(\kappa)d\kappa, \tag{3}
$$

where

$$
\vartheta_{[n]}^{\psi}(\kappa) = \left(\frac{1}{\psi'(\iota)}\frac{d}{d\kappa}\right)^n \vartheta(\kappa) \ \text{ and } \ n = [q]+1,
$$

and $[q]$ denotes the integer part of the real number q.

Proposition 2.4. *[6] Let $q > 0$, $\vartheta \in C^{n-1}(\Lambda, \mathbb{R})$, then we have the following*

1. ${}^{C}\mathfrak{D}_{0^+}^{q,\psi}\mathfrak{I}_{0^+}^{q,\psi}\vartheta(\iota) = \vartheta(\iota)$.
2. $\mathfrak{I}_{0^+}^{q,\psi}{}^{C}\mathfrak{D}_{0^+}^{q,\psi}\vartheta(\iota) = \vartheta(\iota) - \sum_{k=0}^{n-1}\frac{\vartheta_{[k]}^{\psi}(0)}{k!}(\psi(\iota)-\psi(0))^k$.
3. $\mathfrak{I}_{0^+}^{q,\psi}$ *is linear and bounded from $C(\Lambda, \mathbb{R})$ to $C(\Lambda, \mathbb{R})$.*

Definition 2.5. *[22] Let $\vartheta, \psi : [0,\infty) \to \mathbb{R}$ be real valued functions such that $\psi(\iota)$ is continuous and $\psi'(\iota) > 0$ on $[0,\infty)$. The generalized Laplace transform of ϑ is denoted by*

$$
\mathcal{L}_\psi\{\vartheta(\kappa)\}(\mu) = \int_0^\infty e^{-\mu(\psi(\kappa)-\psi(0))}\psi'(\kappa)\vartheta(\kappa)d\kappa. \tag{4}
$$

for all κ.

Definition 2.6. *[22] Let ϑ and σ be two piecewise continuous functions on Λ and $\psi(\iota)$ of exponential order. We define the generalized convolution of ϑ and σ by*

$$
(\vartheta *_\psi \sigma)(\iota) = \int_0^\iota \vartheta(\kappa)\sigma\left(\psi^{-1}(\psi(\iota)+\psi(0)-\psi(\kappa))\right)\psi'(\kappa)d\kappa.
$$

Theorem 2.7. *[22] Let $q > 0$ and ϑ be a piecewise continuous function of exponential order on Λ and $\psi(\kappa)$. Then*

$$
\mathcal{L}_\psi\{I_{0^+}^{q,\psi}\vartheta(\kappa)\}(\mu) = \frac{I_{0^+}^{q,\psi}\vartheta(\kappa)}{\mu^l}. \tag{5}
$$

Theorem 2.8. *[4, 37] Let ϑ_1, ϑ_2 be two integrable functions and $\hbar$ be a continuous function on Λ. Let $\psi \in C(\Lambda, \mathbb{R})$ be an increasing function to the extent that $\psi'(\iota) > 0$ for $\iota \in \Lambda$. Suppose that*

1. *ϑ_1 and ϑ_2 are nonnegative.*
2. *$\hbar$ is nonnegative and nondecreasing. In case*

$$
\vartheta_1(\iota) \le \vartheta_2(\iota) + \hbar(\iota)\int_0^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\vartheta_1(\varsigma)d\varsigma,
$$

subsequently

$$
\vartheta_1(\iota) \le \vartheta_2(\iota) + \int_0^\iota \sum_{w=1}^{\infty}\frac{(\hbar(\iota)\Gamma(l))^w}{\Gamma(wl)}(\psi(\iota)-\psi(\varsigma))^{wq-1}\psi'(\varsigma)\vartheta_2(\varsigma)d\varsigma,
$$

for all $\iota \in \Lambda$.

Corollary 2.9. *Under the hypotheses of Theorem 2.8, let ϑ_2 be nondecreasing function on Λ. Then we have*

$$\vartheta_1(\iota) \leq \vartheta_2(\iota) E_l[\hbar(\iota)\Gamma(l)(\psi(\iota) - \psi(0))^l], \quad \iota \in \Lambda$$

where

$$E_l(\varrho) = \sum_{w=0}^{\infty} \frac{\varrho^w}{\Gamma(wl+1)},$$

is the Mittag-Leffler function with one parameter for all $\varrho \in \mathbb{C}$ and $l > 0$.

Definition 2.10. *[41] The Mittag–Leffler function E_{p_1,q_1} is defined as:*

$$E_{p_1,q_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(p_1 k + q_1)}, \ z, q_1 \in \mathbb{C}, \ R(p_1) > 0,$$

when $p_1 = q_1 = 1$, we can see that $E_{1,1}(z) = e^z$.

Lemma 2.11. *[22] Let $p > 0$, $\vartheta \in AC^n_\psi[0, T]$ for any $T > 0$, and $\vartheta^{[k]}$, $k = 0, 1, \ldots \alpha$, be of $\psi(\iota)$-exponential order. Then,*

$$\mathcal{L}_\psi\{({}^C\mathfrak{D}^{p,\psi}_{0^+}\vartheta)(\iota)\}(\varsigma) = \varsigma^p[\mathcal{L}_\psi\{\vartheta(\iota)\} - \sum_{k=0}^{n-1} \varsigma^{-k-1}(\vartheta^{[k]})(0^+)] \tag{6}$$

Lemma 2.12. *[22] If $R(\varsigma) > 0$, $\xi_1 \in \mathbb{C}$, $|\xi_1 \varsigma^{-p_1}| < 1$, then*

$$\mathcal{L}_\psi\{E_p(\xi(\psi(\iota) - \psi(0))^p\} = \frac{\varsigma^{p-1}}{\varsigma^p - \xi} \tag{7}$$

$$\mathcal{L}_\psi\left\{(\psi(\iota) - \psi(0))^{q-1} E_{p,q}(\xi(\psi(\iota) - \psi(0))^p)\right\}(\varsigma) = \frac{\varsigma^{p-q}}{\varsigma^p - \xi}, \tag{8}$$

3. Main result: Uniqueness and Ulam Hyers Stability

In this fragment, we look for the solution to our problem (1), prove its uniqueness, and then state and prove four types of UH stabilities.

Definition 3.1. *[38] The (p, q)-order coupled system of Volterra integro-differential equations* (1) *are said to be UH stable if there exists $\Theta_{\vartheta,\omega} = \max\{\Theta_\vartheta, \Theta_\omega\} > 0$ such that, for $\varepsilon = \max\{\varepsilon_\vartheta, \varepsilon_\omega\} > 0$ and for every solution $(\overline{\vartheta}, \overline{\omega}) \in \mathcal{X} \times \mathcal{X}$ of the inequality*

$$\begin{cases} \left| {}^C\mathfrak{D}^{p_1,\psi}\overline{\vartheta}(\iota) - \xi_1 {}^C\mathfrak{D}^{q_1,\psi}\overline{\vartheta}(\iota) - \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma)) d\varsigma\right) \right| \leq \varepsilon_\vartheta, & \iota \in \Lambda, \\ \left| {}^C\mathfrak{D}^{p_2,\psi}\overline{\omega}(\iota) - \xi_2 {}^C\mathfrak{D}^{q_2,\psi}\overline{\omega}(\iota) - \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma)) d\varsigma\right) \right| \leq \varepsilon_\omega, & \iota \in \Lambda, \end{cases} \tag{9}$$

there exists a unique solution $(\vartheta, \omega) \in \mathcal{X} \times \mathcal{X}$ with

$$\left\| (\vartheta, \omega) - (\overline{\vartheta}, \overline{\omega}) \right\| \leq \Theta_{\vartheta,\omega}\varepsilon, \quad \iota \in \Lambda.$$

Definition 3.2. *[38] The (p,q)-order coupled system of Volterra integro-differential equations* (1) *are said to be generalized UH stable if there exists $\rho \in C(\Lambda, \mathbb{R}^+)$ with $\rho(0) = 0$ such that, for every solution $(\overline{\chi}, \overline{\zeta}) \in \mathcal{X} \times \mathcal{X}$ of the inequality* (9) *there exists a unique solution $(\vartheta, \omega) \in \mathcal{X} \times \mathcal{X}$ of* (1) *satisfying*

$$\left\|(\vartheta, \omega) - (\overline{\vartheta}, \overline{\omega})\right\| \leq \rho(\varepsilon), \qquad \iota \in \Lambda.$$

Denote $\Phi_{\vartheta,\omega} = \max\{\Phi_\vartheta, \Phi_\omega\} \in C(\Lambda, \mathbb{R}^+)$ and $\Theta_{\Phi_\vartheta,\Phi_\omega} = \max\{\Theta_{\Phi_\vartheta}, \Theta_{\Phi_\omega}\} > 0$.

Definition 3.3. *[38] The (p,q)-order coupled system of Volterra integro-differential equations* (1) *are said to be UH-Rassias stable with respect to $\Phi_{\vartheta,\omega}$ if there exists a constant $\Theta_{\Phi_\vartheta,\Phi_\omega}$ such that, for some $\varepsilon = \max\{\varepsilon_\vartheta, \varepsilon_\omega\} > 0$ and for any approximate solution $(\overline{\vartheta}, \overline{\omega}) \in \mathcal{X} \times \mathcal{X}$ of the inequality*

$$\begin{cases} \left|{}^C\mathfrak{D}^{p_1,\psi}\overline{\vartheta}(\iota) - \xi_1 {}^C\mathfrak{D}^{q_1,\psi}\overline{\vartheta}(\iota) - \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right)\right| \leq \Phi_\vartheta(\iota)\varepsilon_\vartheta, \qquad \iota \in \Lambda, \\ \left|{}^C\mathfrak{D}^{p_2,\psi}\overline{\omega}(\iota) - \xi_2 {}^C\mathfrak{D}^{q_2,\psi}\overline{\omega}(\iota) - \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right)\right| \leq \Phi_\omega(\iota)\varepsilon_\omega, \qquad \iota \in \Lambda, \end{cases} \tag{10}$$

there exists a unique solution $(\vartheta, \omega) \in \mathcal{X} \times \mathcal{X}$ with

$$\left\|(\vartheta, \omega) - (\overline{\vartheta}, \overline{\omega})\right\| \leq \Theta_{\vartheta,\omega}\Phi_{\vartheta,\omega}(\iota)\varepsilon, \qquad \iota \in \Lambda.$$

Definition 3.4. *[38] The (p,q)-order coupled system of Volterra integro-differential equations* (1) *are said to be generalized UH-Rassias stable with respect to $\Phi_{\vartheta,\omega}$ if there exists a constant $\Theta_{\Phi_\vartheta,\Theta_\omega}$ such that, for any approximate solution $(\overline{\vartheta}, \overline{\omega}) \in \mathcal{X} \times \mathcal{X}$ of the inequality* (10)*, there exists a unique solution $(\vartheta, \omega) \in \mathcal{X} \times \mathcal{X}$ of* (1) *satisfying*

$$\left\|(\vartheta, \omega) - (\overline{\vartheta}, \overline{\omega})\right\| \leq \Theta_{\vartheta,\omega}\Phi_{\vartheta,\omega}(\iota)\varepsilon, \qquad \iota \in \Lambda.$$

Remark 3.5. *We say that $(\overline{\vartheta}, \overline{\omega}) \in \mathcal{X} \times \mathcal{X}$ is a solution of the system of inequalities* (9) *if there exist functions $\varrho_\tau, \varrho_\kappa \in C(\Lambda, \mathbb{R})$ depending upon ϑ, ω respectively, such that*

1) $|\varrho_\tau(\iota)| \leq \varepsilon_\vartheta, |\varrho_\kappa(\iota)| \leq \varepsilon_\omega, \qquad \iota \in \Lambda,$
2)
$$\begin{cases} {}^C\mathfrak{D}^{p_1,\psi}\overline{\vartheta}(\iota) - \xi_1 {}^C\mathfrak{D}^{q_1,\psi}\overline{\vartheta}(\iota) = \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right) + \varrho_\tau(\iota), \\ {}^C\mathfrak{D}^{p_2,\psi}\overline{\omega}(\iota) - \xi_2 {}^C\mathfrak{D}^{q_2,\psi}\overline{\omega}(\iota) = \hbar\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right) + \varrho_\kappa(\iota), \qquad \iota \in \Lambda. \end{cases}$$

We shall prove our results concerning the (p,q)-order coupled system of Volterra integro-differential equations (1) under the following assumptions:

$(\mathcal{A}_1)$ The function $\hbar : \Lambda \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfies:

- (i) $\hbar(\cdot, \vartheta, \omega) : \Lambda \to \mathcal{X}$ is measurable for all $(\vartheta, \omega) \in \mathcal{X} \times \mathcal{X}$ and $\hbar(\iota, \cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous for a.e. $\iota \in \Lambda$.
- (ii) There exist functions $\eta(\iota)$, $\lambda_1(\cdot) \in L^{\frac{1}{\delta}}(\Lambda, \mathbb{R}^+)$, $\delta \in (0, p)$ and a continuous function $\lambda_2(\cdot)$ such that

$$\|\hbar(\iota, \vartheta, \omega)\| \leq \eta(\iota) + \lambda_1(\iota)\|\vartheta\| + \lambda_2(\iota)\|\omega\|, \ \textit{for a.e. } \iota \in \Lambda, \ \forall \vartheta, \omega \in \mathcal{X}.$$

- (iii) There exist constants $\ell_1, \ell_2 > 0$, such that

$$\|\hbar(\iota, \vartheta_1, \omega_1) - \hbar(\iota, \vartheta_2, \omega_2)\| \leq \ell_1\|\vartheta_1 - \vartheta_2\| + \ell_2\|\omega_1 - \omega_2\|, \ \textit{for a.e. } \iota \in \Lambda, \ \forall \vartheta_j, \omega_j \in \mathcal{X}, \ j = 1, 2.$$

($\mathcal{A}_2$) For each $(\iota,\varsigma) \in D = \{(\iota,\varsigma) \in \Lambda\times\Lambda, \iota \geq \varsigma\}$, the function $g(\iota,\varsigma,\cdot) : \mathcal{X} \to \mathcal{X}$ is continuous and for each $\omega \in \mathcal{X}$, $g(\cdot,\cdot,\omega) : D \to \mathcal{X}$ is strongly measurable. Moreover, there exists a function $K : D \to \mathbb{R}^+$ with $\int_0^\iota K(\iota,\varsigma)ds := k^*(\iota) \in L^\infty(\Lambda)$ such that

$$\|g(\iota,\varsigma,\omega)\| \leq K(\iota,\varsigma)\|\omega\|, \ for\ a.e.\ (\iota,\varsigma) \in D,\ \forall\omega \in \mathcal{X},$$

and

$$\|g(\iota,\varsigma,\vartheta) - g(\iota,\varsigma,\omega)\| \leq K(\iota,\varsigma)\|\vartheta-\omega\|, \ for\ a.e.\ (\iota,\varsigma) \in D,\ \forall\vartheta,\omega \in \mathcal{X}.$$

The following lemma examines the initial version of the problem previously outlined in (1).

Lemma 3.6. *Let $\hbar$ be a continuous and linear function. The solution of*

$$\begin{cases} {}^C\mathfrak{D}^{p,\psi}\vartheta(\iota) = \xi\ {}^C\mathfrak{D}^{q,\psi}\vartheta(\iota) + \hbar(\iota), \quad \iota \in \Lambda, \\ {}^C\mathfrak{D}^{k,\psi}\vartheta(0) = \vartheta_0; \quad k = 0,...,\alpha-1 \end{cases} \tag{11}$$

is given as:

$$\begin{aligned} \vartheta(\iota) &= \sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^k E_{p-q,k+1}(\xi(\psi(\iota)-\psi(0))^{p-q})\vartheta_0 \\ &- \xi\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p-q+k}E_{p-q,p-q+k+1}(\xi(\psi(\iota)-\psi(0))^{p-q})\vartheta_0 \\ &+ \int_0^\iota \psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p-1}E_{p-q,p}(\xi(\psi(\iota)-\psi(\varsigma))^{p-q})\hbar(\varsigma,\vartheta(\varsigma))d\varsigma, \end{aligned} \tag{12}$$

where $\xi \in \mathbb{R}$, $\alpha - 1 < p \leq \alpha$ and $\beta - 1 < q \leq \beta$ as $\alpha,\beta \in \mathbb{Z}^+$ and $\beta \leq \alpha$.

Proof. Applying the generalized Laplace transform to both sides of Equation (11) and utilizing Lemma 2.11, we derive

$$\begin{aligned} \mathcal{L}_\psi\{\hbar(\iota)\}(\varsigma) &= \mathcal{L}_\psi\{{}^C\mathfrak{D}^{p,\psi}\vartheta(\iota) - \xi\ {}^C\mathfrak{D}^{q,\psi}\vartheta(\iota)\}(\varsigma) \\ &= \mathcal{L}_\psi\{{}^C\mathfrak{D}^{p,\psi}\vartheta(\iota)\}(\varsigma) - \xi\mathcal{L}_\psi\{{}^C\mathfrak{D}^{q,\psi}\vartheta(\iota)\}(\varsigma) \\ &= \varsigma^p\mathcal{L}_\psi\{\vartheta(\iota)\}(\varsigma) - \sum_{k=0}^{\alpha-1}\varsigma^{p-1-k}\vartheta_0 - \xi\varsigma^q\mathcal{L}_\psi\{\vartheta(\iota)\}(\varsigma) + \xi\sum_{k=0}^{\beta-1}\varsigma^{q-k-1}\vartheta_0, \end{aligned}$$

then

$$(\varsigma^p - \xi\varsigma^q)\mathcal{L}_\psi\{\vartheta(\iota)\}(\varsigma) = \sum_{k=0}^{\alpha-1}\varsigma^{p-1-k}\vartheta_0 - \xi\sum_{k=0}^{\beta-1}\varsigma^{q-k-1}\vartheta_0 + \mathcal{L}_\psi\{\hbar(\iota)\}(\varsigma),$$

which means that

$$\mathcal{L}_\psi\{\vartheta(\iota)\}(\varsigma) = \frac{\sum_{k=0}^{\alpha-1}\iota^{p-1-k}\chi_0 - \xi\sum_{k=0}^{\beta-1}\iota^{q-k-1}\chi_0 + \mathcal{L}_\psi\{\hbar(\iota)\}(\varsigma)}{(\varsigma^p - \xi\varsigma^q)}. \tag{13}$$

By applying the inverse generalized Laplace transform to both sides of equation (13), we obtain:

$$\begin{aligned} \vartheta(\iota) = \mathcal{L}_\psi^{-1}\{\mathcal{L}_\psi\{\vartheta(\iota)\}(\varsigma)\} &= \mathcal{L}_\psi^{-1}\left\{\frac{\sum_{k=0}^{\alpha-1}\varsigma^{p-1-k}\vartheta_0 - \xi\sum_{k=0}^{\beta-1}\varsigma^{q-k-1}\vartheta_0 + \mathcal{L}_\psi\{\hbar(\iota)\}(\varsigma)}{(\varsigma^p - \xi\varsigma^q)}\right\} \\ &= \sum_{k=0}^{\alpha-1}\mathcal{L}_\psi^{-1}\left\{\frac{\varsigma^{p-1-k}}{(\varsigma^p - \xi\varsigma^q)}\right\}\vartheta_0 - \xi\sum_{k=0}^{\beta-1}\mathcal{L}_\psi^{-1}\left\{\frac{\varsigma^{q-k-1}}{(\varsigma^p - \xi\varsigma^q)}\right\}\vartheta_0 + \mathcal{L}_\psi^{-1}\left\{\frac{1}{(\varsigma^p - \xi\varsigma^q)}\right\} \star \hbar(\iota). \end{aligned} \tag{14}$$

Lemma 2.12 is useful for determining certain inverse Laplace transforms.

$$\mathcal{L}_{\psi}^{-1}\left\{\frac{\varsigma^{p-1-k}}{(\varsigma^{p}-\xi\varsigma^{q})}\right\}=(\psi(\iota)-\psi(0))^{k}E_{p-q,k+1}(\xi(\psi(\iota)-\psi(0))^{p-q}). \tag{15}$$

$$\mathcal{L}_{\psi}^{-1}\left\{\frac{\varsigma^{q-k-1}}{(\varsigma^{p}-\xi\varsigma^{q})}\right\}=(\psi(\iota)-\psi(0))^{p-q+k}E_{p-q,p-q+k+1}(\xi(\psi(\iota)-\psi(0))^{p-q}). \tag{16}$$

$$\mathcal{L}_{\psi}^{-1}\left\{\frac{1}{(\varsigma^{p}-\xi\varsigma^{q})}\right\}=(\psi(\iota)-\psi(0))^{p-1}E_{p-q,p}(\xi(\psi(\iota)-\psi(0))^{p-q}). \tag{17}$$

Substituting (15), (16) and (17) in (14), we get

$$\begin{aligned}\vartheta(\iota)&=\sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^{k}E_{p-q,k+1}(\xi(\psi(\iota)-\psi(0))^{p-q})\vartheta_0\\&-\xi\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p-q+k}E_{p-q,p-q+k+1}(\xi(\psi(\iota)-\psi(0))^{p-q})\vartheta_0\\&+(\psi(\iota)-\psi(\varsigma))^{p-1}E_{p-q,p}(\xi(\psi(\iota)-\psi(\varsigma))^{p-q})\star\hbar(\iota).\end{aligned}$$

□

From Lemma 3.6, we have demonstrated that the pair $(\vartheta,\omega)\in C$ serves as the solution to the system presented in (1), where

$$\begin{aligned}\vartheta(\iota)&=\sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^{k}E_{p_1-q_1,k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0\\&-\xi_1\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_1-q_1,p_1-q_1+k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0\\&+\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_1-1}E_{p_1-q_1,p_1}(\xi_1(\psi(\iota)-\psi(\varsigma))^{p_1-q_1})\hbar\left(\varsigma,\vartheta(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\omega(\sigma))d\sigma\right)d\varsigma,\end{aligned} \tag{18}$$

and

$$\begin{aligned}\omega(\iota)&=\sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^{k}E_{p_2-q_2,k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0\\&-\xi_2\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_2-q_2,p_2-q_2+k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0\\&+\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_2-1}E_{p_2-q_2,p_2}(\xi_2(\psi(\iota)-\psi(\varsigma))^{p_2-q_2})\hbar\left(\varsigma,\vartheta(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\omega(\sigma))d\sigma\right)d\varsigma.\end{aligned} \tag{19}$$

We define the operator $\mathbb{G}_i : C\to C$, by $i=1,2$

$$\mathbb{G}(\vartheta,\omega)=\{\mathbb{G}_1(\vartheta,\omega),\mathbb{G}_2(\vartheta,\omega)\},$$

with

$$\begin{aligned}
&\mathbb{G}_1(\vartheta,\omega)(\iota)\\
&=\sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^k E_{p_1-q_1,k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0\\
&-\xi_1\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_1-q_1,p_1-q_1+k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0\\
&+\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_1-1}E_{p_1-q_1,p_1}(\xi_1(\psi(\iota)-\psi(\varsigma))^{p_1-q_1})\hbar\left(\varsigma,\vartheta(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\omega(\sigma))d\sigma\right)d\varsigma,
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&\mathbb{G}_2(\vartheta,\omega)(\iota)\\
&=\sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^k E_{p_2-q_2,k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0\\
&-\xi_2\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_2-q_2,p_2-q_2+k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0\\
&+\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_2-1}E_{p_2-q_2,p_2}(\xi_2(\psi(\iota)-\psi(\varsigma))^{p_2-q_2})\hbar\left(\varsigma,\vartheta(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\omega(\sigma))d\sigma\right)d\varsigma,
\end{aligned}\tag{21}$$

then $\mathbb{G}(\vartheta,\omega)(\iota)$ is solutions of the problem (1).

Theorem 3.7. *If assumptions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are satisfied, then there exists a unique solution of the (p,q)-order coupled system of Volterra integro-differential equations* (1) *on Λ, if and only if*

$$\begin{aligned}
&(\ell_1+\ell_2\|k^*\|)\Bigg[\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\\
&+\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\Bigg]<1
\end{aligned}\tag{22}$$

Additionally, with

$$\begin{aligned}
\Sigma=1-\Bigg[&\frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1}\\
&\times\frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}\Bigg].
\end{aligned}\tag{23}$$

this unique solution is UH-stable and consequently generalized UH-stable.

Proof. The proof of this theorem is organized into two parts. In the first part, we will establish that the (p,q)-order coupled system of Volterra integro-differential equations (1) has a unique solution by employing the Banach contraction principle. The second part will address Ulam Hyers' Stability of solutions related to the problem above.

Part I. Let $\vartheta, \omega, \overline{\vartheta}, \overline{\omega} \in C$, by using the condition $\mathcal{A}_2$:

$$
\begin{aligned}
&\left\|\mathbb{G}_1(\vartheta,\omega) - \mathbb{G}_1(\overline{\vartheta},\overline{\omega})\right\|_{\mathcal{X}} \\
&\leq \left\| \int_0^{\iota} \psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_1-1} E_{p_1-q_1,p_1}(\xi_1(\psi(\iota)-\psi(\varsigma))^{p_1-q_1}) \right. \\
&\times \left. \left[\hbar\left(\varsigma,\vartheta(\varsigma),\int_0^{\varsigma} g(\varsigma,\sigma,\omega(\sigma))d\sigma\right) - \hbar\left(\varsigma,\overline{\vartheta}(\varsigma),\int_0^{\varsigma} g(\varsigma,\sigma,\overline{\omega}(\sigma))d\varsigma\right)\right]d\varsigma \right\|_{\mathcal{X}} \\
&\leq \sup_{\iota\in\Lambda} \left| \sum_{k=0}^{\infty} \int_0^{\iota} \frac{\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{(p_1-q_1)k+p_1-1}}{\Gamma((p_1-q_1)k+p_1)} \xi_1^k \right| \\
&\times \left[\ell_1 \left\|\vartheta(\varsigma)-\overline{\vartheta}(\varsigma)\right\| + \ell_2 \int_0^{\varsigma} I(\varsigma,\sigma)\left\|\omega(\sigma)-\overline{\omega}(\sigma)\right\| d\sigma\right] d\varsigma \\
&\leq \sup_{\iota\in\Lambda} \sum_{k=0}^{\infty} \frac{(\psi(\iota)-\psi(0))^{(p_1-q_1)k+p_1}}{\Gamma((p_1-q_1)k+p_1+1)} \xi_1^k \left[\ell_1\left\|\vartheta-\overline{\vartheta}\right\| + \ell_2\|i^*\|\,\|\omega-\overline{\omega}\|\right] d\varsigma \\
&\leq \sup_{\iota\in\Lambda} (\psi(\iota)-\psi(0))^{p_1} E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})(\ell_1+\ell_2\|i^*\|)\left\|(\vartheta,\omega)-(\overline{\vartheta},\overline{\omega})\right\|,
\end{aligned}
$$

By using the same approach, we achieve the same result:

$$
\begin{aligned}
&\left\|\mathbb{G}_2(\vartheta,\omega) - \mathbb{G}_2(\overline{\vartheta},\overline{\omega})\right\|_{\mathcal{X}} \\
&\leq \sup_{\iota\in\Lambda} (\psi(\iota)-\psi(0))^{p_2} E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})(\ell_1+\ell_2\|j^*\|)\left\|(\vartheta,\omega)-(\overline{\vartheta},\overline{\omega})\right\|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left\|\mathbb{G}(\vartheta,\omega) - \mathbb{G}(\overline{\vartheta},\overline{\omega})\right\|_{\mathcal{X}} \\
&\leq \left\|\mathbb{G}_2(\vartheta,\omega) - \mathbb{G}_2(\overline{\vartheta},\overline{\omega})\right\|_{\mathcal{X}} + \left\|\mathbb{G}_2(\vartheta,\omega) - \mathbb{G}_2(\overline{\vartheta},\overline{\omega})\right\|_{\mathcal{X}} \\
&\leq [\sup_{\iota\in\Lambda} (\psi(\iota)-\psi(0))^{p_1} E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1}) \\
&+ \sup_{\iota\in\Lambda} (\psi(\iota)-\psi(0))^{p_2} E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})](\ell_1+\ell_2\|k^*\|)\left\|(\vartheta,\omega)-(\overline{\vartheta},\overline{\omega})\right\|,
\end{aligned}
$$

where $k^* = \sup_{\iota\in\Lambda}\{i^*, j^*\}$. It is clear that the mapping $\mathbb{G}$ is a contraction, as evidenced by condition (22).

By applying the well-known Banach fixed-point theorem, we can confirm the existence of a unique fixed point for $\mathbb{G}$. This result definitively establishes that the system described by equation (1) has a unique solution within the domain Λ.

Part II. Now, we discuss the UH stability of the unique solution of the (p,q)-order coupled system of Volterra integro-differential equations (1).

Let's take a look at the following system:

$$
\begin{cases}
{}^{C}\mathfrak{D}^{p_1,\psi}\overline{\vartheta}(\iota) - \xi_1{}^{C}\mathfrak{D}^{q_1,\psi}\overline{\vartheta}(\iota) = \hbar\left(\iota,\overline{\vartheta}(\iota),\int_0^{\iota} g(\iota,\varsigma,\overline{\omega}(\varsigma))d\varsigma\right) + \varrho_{\tau}(\iota), \\
{}^{C}\mathfrak{D}^{p_2,\psi}\overline{\omega}(\iota) - \xi_2{}^{C}\mathfrak{D}^{q_2,\psi}\overline{\omega}(\iota) = \hbar\left(\iota,\overline{\vartheta}(\iota),\int_0^{\iota} g(\iota,\varsigma,\overline{\omega}(\varsigma))d\varsigma\right) + \varrho_{\kappa}(\iota), \qquad \iota \in \Lambda.
\end{cases}
\tag{24}
$$

By (18), (19) the solution of the problem (24) is

$$\begin{cases} \overline{\vartheta}(\iota) = \sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^{k}E_{p_1-q_1,k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0 \\ -\xi_1\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_1-q_1,p_1-q_1+k+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\vartheta_0 \\ +\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_1-1}E_{p_1-q_1,p_1}(\xi_1(\psi(\iota)-\psi(\varsigma))^{p_1-q_1}) \\ \times\Big[\hbar\Big(\varsigma,\overline{\vartheta}(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\overline{\omega}(\sigma))d\sigma\Big)+\varrho_\tau(\varsigma)\Big]d\varsigma, \\ \overline{\omega}(\iota) = \sum_{k=0}^{\alpha-1}(\psi(\iota)-\psi(0))^{k}E_{p_2-q_2,k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0 \\ -\xi_2\sum_{k=0}^{\beta-1}(\psi(\iota)-\psi(0))^{p_1-q_1+k}E_{p_2-q_2,p_2-q_2+k+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\omega_0 \\ +\int_0^{\iota}\psi'(\varsigma)(\psi(\iota)-\psi(\varsigma))^{p_2-1}E_{p_2-q_2,p_2}(\xi_2(\psi(\iota)-\psi(\varsigma))^{p_2-q_2}) \\ \times\Big[\hbar\Big(\varsigma,\overline{\vartheta}(\varsigma),\int_0^{\varsigma}g(\varsigma,\sigma,\overline{\omega}(\sigma))d\sigma\Big)+\varrho_\kappa(\varsigma)\Big]d\varsigma. \end{cases} \tag{25}$$

With the help of **Part I.**, we consider

$$\left\|\vartheta-\overline{\vartheta}\right\| \le \sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})[\ell_1\left\|\vartheta-\overline{\vartheta}\right\|+\ell_2\|i^*\|\,\|\omega-\overline{\omega}\|+\varepsilon_\vartheta]. \tag{26}$$

$$\|\omega-\overline{\omega}\| \le \sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})[\ell_1\left\|\vartheta-\overline{\vartheta}\right\|+\ell_2\|j^*\|\,\|\omega-\overline{\omega}\|+\varepsilon_\omega]. \tag{27}$$

From (26) and (27), we have

$$\begin{cases} \left\|\vartheta-\overline{\vartheta}\right\| - \dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1}\|\omega-\overline{\omega}\| \\ \le \varepsilon_\vartheta\dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1}. \\ \|\omega-\overline{\omega}\| - \dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}\left\|\vartheta-\overline{\vartheta}\right\| \\ \le \varepsilon_\omega\dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}. \end{cases}$$

Let $\Theta_\vartheta = \dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1}$,

and $\Theta_\omega = \dfrac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}$, then the last two inequalities can be written in matrix form as

$$\begin{bmatrix} 1 & \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1} \\ \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \left\|\vartheta-\overline{\vartheta}\right\| \\ \|\omega-\overline{\omega}\| \end{bmatrix} \leq \begin{bmatrix} \Theta_\vartheta\varepsilon_\vartheta \\ \Theta_\omega\varepsilon_\omega \end{bmatrix}.$$

Then

$$\begin{bmatrix} \left\|\vartheta-\overline{\vartheta}\right\| \\ \|\omega-\overline{\omega}\| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Sigma} & \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1)} \\ \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|)} & \frac{1}{\Sigma} \end{bmatrix} \times \begin{bmatrix} \Theta_\vartheta\varepsilon_\vartheta \\ \Theta_\omega\varepsilon_\omega \end{bmatrix}. \tag{28}$$

Where

$$\Sigma = 1 - \left[\frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1} \times \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}\right].$$

From system above, we have

$$\left\|\vartheta-\overline{\vartheta}\right\| \leq \frac{\Theta_\vartheta\varepsilon_\vartheta}{\Sigma} + \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1)},$$

and

$$\|\omega-\overline{\omega}\| \leq \frac{\Theta_\omega\varepsilon_\omega}{\Sigma} + \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|)}.$$

This suggests that

$$\begin{aligned} \left\|\vartheta-\overline{\vartheta}\right\| + \|\omega-\overline{\omega}\| \leq{}& \frac{\Theta_\vartheta\varepsilon_\vartheta}{\Sigma} + \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1)} \\ &+ \frac{\Theta_\omega\varepsilon_\omega}{\Sigma} + \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|)}. \end{aligned}$$

If $\max\{\varepsilon_\vartheta, \varepsilon_\omega\} = \varepsilon$ and

$$\Theta_{\vartheta,\omega} = \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1)} + \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{\Sigma(1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|)},$$

then

$$\left\|(\vartheta,\omega)-(\overline{\vartheta},\overline{\omega})\right\| \le \Theta_{\vartheta,\omega}\varepsilon.$$

This indicates that the system described by equation (1) is uniformly stable. Additionally, if

$$\left\|(\vartheta,\omega)-(\overline{\vartheta},\overline{\omega})\right\| \le \Theta_{\vartheta,\omega}\rho(\varepsilon).$$

with $\rho(0) = 0$, consequently, the generalized HU stable solution to (1) is found. □

Theorem 3.8. *If assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *are satisfied and*

$$\Sigma = 1 - \left[\frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_2\|i^*\|}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\ell_1} \times \frac{\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_1}{1-\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\ell_2\|j^*\|}\right] > 0,$$

then the unique solution of the (p,q)*-order coupled system of Volterra integro-differential equations* (1) *is UH-Rassias stable and generalized UH-Rassias stable.*

Proof. Using Definition 3.3 and 3.4, we can get our result by performing the same steps as in **Part II.** in the proof of Theorem 3.7. □

4. Numerical example

The final section examines an example that supports our main result. Consider the following fractional differential equation:

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{7}{2}}\vartheta(\iota,y) - \frac{3}{18}\,{}^{C}\mathfrak{D}^{\frac{5}{2}}\vartheta(\iota,y) = e^{-3\iota} + \left[\frac{1}{(\iota+3)^2}\cos(\vartheta(\iota)) + \frac{t^2}{5}\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota}d\varsigma\right](y), \\ {}^{C}\mathfrak{D}^{\frac{10}{3}}\omega(\iota,y) - \frac{3}{18}\,{}^{C}\mathfrak{D}^{\frac{8}{3}}\omega(\iota,y) = e^{-2\iota} + \left[\frac{e^{-\iota}}{8+e^{\iota}}\arctan(\vartheta(\iota)) + \frac{t^2}{5}\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota^2+1}d\varsigma\right](y), \\ \vartheta(\iota,0) = \vartheta'(\iota,0) = \vartheta''(\iota,0) = \vartheta'''(\iota,0) = \omega(\iota,0) = \omega'(\iota,0) = \omega''(\iota,0) = \omega'''(\iota,0) = 1. \end{cases} \tag{29}$$

Where $\psi(\iota) = e^\iota$, $p_1 = \frac{7}{2}$, $p_2 = \frac{10}{3}$, $q_1 = \frac{5}{2}$, $q_2 = \frac{8}{3}$, $T = 1$, $\xi_1 = \xi_2 = \frac{3}{18}$,

$$\hbar_1\left(\iota,\vartheta(\iota),\int_0^\iota g_1(\iota,\varsigma,\omega(\varsigma))d\varsigma\right)(y) = e^{-3\iota} + \left[\frac{1}{(\iota+3)^2}\cos(\vartheta(\iota)) + \frac{t^2}{5}\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota}d\varsigma\right](y),$$

$$\hbar_2\left(\iota, \vartheta(\iota), \int_0^\iota g_2(\iota, \varsigma, \omega(\varsigma))d\varsigma\right)(y) = e^{-2\iota} + \left[\frac{e^{-\iota}}{8+e^\iota}\arctan(\vartheta(\iota)) + \frac{t^2}{5}\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota^2+1}d\varsigma\right](y),$$

We have

$$\begin{aligned}\left\|\hbar_1\left(\iota, \vartheta(\iota), \int_0^\iota g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma\right)\right\| &= e^{-3\iota} + \frac{1}{(\iota+3)^2}\|\vartheta(\iota)\| + \frac{t^2}{5}\left\|\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota}d\varsigma\right\| \\ &:= \mu_1(\iota) + \lambda_{1,1}(\iota)\|\vartheta(\iota)\| + \lambda_{1,2}(\iota)\left\|\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota}d\varsigma\right\|\end{aligned}$$

and

$$\begin{aligned}\left\|\hbar_2\left(\iota, \vartheta(\iota), \int_0^\iota g_2(\iota, \varsigma, \omega(\varsigma))d\varsigma\right)\right\| &= e^{-2\iota} + \frac{e^{-\iota}}{8+e^\iota}\|\vartheta(\iota)\| + \frac{t^2}{5}\left\|\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota^2+1}d\varsigma\right\| \\ &:= \mu_2(\iota) + \lambda_{2,1}(\iota)\|\vartheta(\iota)\| + \lambda_{2,2}(\iota)\left\|\int_0^\iota \varsigma^2\cos\frac{\omega(\varsigma)}{\iota^2+1}d\varsigma\right\|\end{aligned}$$

.

For any $\vartheta, \overline{\vartheta}, \omega, \overline{\omega} \in \mathcal{X}$,

$$\begin{aligned}&\left\|\hbar_1\left(\iota, \vartheta(\iota), \int_0^\iota g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma\right) - \hbar_1\left(\iota, \overline{\vartheta}(\iota), \int_0^\iota g_1(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right)\right\| \\ &\leq \frac{1}{9}\|\vartheta - \overline{\vartheta}\| + \frac{1}{5}\left\|\int_0^\iota g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma - \int_0^\iota g_1(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right\|,\end{aligned}$$

$$\left\|g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma\right\| \leq I(\iota, \varsigma)\|\vartheta(\varsigma)\|,$$

$$\left\|\int_0^\iota g_1(\iota, \varsigma, \omega(\varsigma))d\varsigma - \int_0^\iota g_1(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right\| \leq I(\iota, \varsigma)\|\omega - \overline{\omega}\|,$$

where

$$\int_0^\iota I(\iota, \varsigma)d\varsigma := i^*(\iota) \in L^\infty([0,1]),$$

and

$$ess\sup_{\iota\in[0,1]}\int_0^\iota I(\iota, \varsigma)d\varsigma = ess\sup_{\iota\in[0,1]}\int_0^\iota \frac{\varsigma^2}{\iota}d\varsigma = \frac{1}{3} := \|i^*\|_{L^\infty},$$

we do the same for g_2,we get

$$\left\|g_2(\iota, \varsigma, \omega(\varsigma))d\varsigma\right\| \leq J(\iota, \varsigma)\|\vartheta(\varsigma)\|,$$

$$\left\|\int_0^\iota g_2(\iota, \varsigma, \omega(\varsigma))d\varsigma - \int_0^\iota g_2(\iota, \varsigma, \overline{\omega}(\varsigma))d\varsigma\right\| \leq J(\iota, \varsigma)\|\omega - \overline{\omega}\|,$$

where

$$\int_0^\iota J(\iota, \varsigma)d\varsigma := j^*(\iota) \in L^\infty([0,1]),$$

and

$$ess \sup_{\iota\in[0,1]} \int_0^\iota J(\iota,\varsigma)d\varsigma = ess \sup_{\iota\in[0,1]} \int_0^\iota \frac{\varsigma^2}{\iota^2+1}d\varsigma = \frac{1}{6} := \|j^*\|_{L^\infty}.$$

Finally we find that, $k^* = \max_{\iota\in[0,1]}\{i^*, j^*\} = \frac{1}{3}$, $\ell_1 = \frac{1}{9}$ and $\ell_2 = \frac{1}{5}$.

Therefore, the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are verified,

$$\begin{aligned}
&(\ell_1 + \ell_2\|k^*\|)\Big[\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_1}E_{p_1-q_1,p_1+1}(\xi_1(\psi(\iota)-\psi(0))^{p_1-q_1})\\
&+\sup_{\iota\in\Lambda}(\psi(\iota)-\psi(0))^{p_2}E_{p_2-q_2,p_2+1}(\xi_2(\psi(\iota)-\psi(0))^{p_2-q_2})\Big]\\
&= (\frac{1}{9}+\frac{1}{5}\times\frac{1}{3})\Big[\sup_{\iota\in\Lambda}(e-1)^{\frac{7}{2}}E_{1,\frac{9}{2}}(\frac{3}{18}(e-1)^1 + \sup_{\iota\in\Lambda}(e-1)^{\frac{10}{3}}E_{\frac{2}{3},\frac{13}{3}}(\frac{3}{18}(e-1)^{\frac{2}{3}}\Big]\\
&= (\frac{1}{9}+\frac{1}{5}\times\frac{1}{3})(0.59895+1.124)\\
&= 0.3063 < 1.
\end{aligned}$$

As a result, (22) is satisfied. As a result, problem (1) has at least one mild solution in $[0,1]$, according to Theorem 3.7.

5. Conclusions

This paper addresses the coupled system of Volterra integro-differential equations of order (p,q).The novelty of the problem considered is that it has been investigated under the ψ-Caputo fractional derivatives, which is more general than the works based on well-known fractional derivatives such as (Caputo fractional derivative, Caputo–Hadamard fractional derivative and Caputo–Katugampola fractional derivative) for different values of the function ψ. The generalized Laplace transform method is employed to determine the solution, followed by the establishment of uniqueness results using Banach's fixed-point theorem. Additionally, various types of stability are investigated in the Ulam-Hyers (UH) sense for the given problems. Finally, a concrete application is presented to demonstrate the effectiveness of the main results. As a future research direction for this paper, we aim to extend these results to investigate the case of the ψ-Hilfer fractional derivative.

Acknowledgements

The authors would like to thank the referees for the valuable comments and suggestions that improve the quality of our paper.

Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts Of Interest

The authors declare that they have no conflict of interest.

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