Filomat 39:9 (2025), 3083–3097
https://doi.org/10.2298/FIL2509083R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New directions in fixed point theory in multiplicative metric spaces

Stojan Radenović[a], Bessem Samet[b,*]

[a]*Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia*
[b]*Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia*

Abstract. The notion of a multiplicative metric space was introduced in 2008 by Bashirov et al. In a such space, the usual triangular inequality is replaced by a multiplicative triangle inequality. The literature includes numerous fixed point results in multiplicative metric spaces. Unfortunately, it was shown that the most obtained fixed point results in such spaces are equivalent to the corresponding fixed point results in metric spaces. In this paper, we open new directions in fixed point theory in multiplicative metric spaces by investigating new contractions on such spaces that cannot be reduced to contractions on metric spaces. We first establish a new multiplicative version of Banach's fixed point theorem. Next, a new multiplicative version of Kannan's fixed point theorem is proved. Unlike Kannan's contraction in metric spaces, we show that a multiplicative contraction of Kannan-type may have more than one fixed point. We also provide sufficient conditions under which any multiplicative contraction of Kannan-type possesses one and only one fixed point. Some examples are provided to illustrate the validity of our obtained results.

1. Introduction

The topic of multiplicative calculus was introduced by Grossman and Katz [12] in 1972. Inspired from [12], Bashirov et al. [5] introduced the notion of multiplicative metric spaces. We recall below some basic definitions related to such spaces.

Let M be a nonempty set. A mapping $D : M \times M \to [1, \infty)$ is called a multiplicative metric, if for all $u, v, w \in M$, we have

(D_1) $D(u, v) = 1$ if and only if $u = v$;

(D_2) $D(u, v) = D(v, u)$;

(D_3) $D(u, v) \leq D(u, w)D(w, v)$ (multiplicative triangle inequality).

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. Multiplicative metric space, metric space, fixed point, existence, uniqueness.
Received: 29 June 2024; Accepted: 11 January 2025
Communicated by Adrian Petruşel
The second author is supported by Researchers Supporting Project number (RSP2024R4), King Saud University, Riyadh, Saudi Arabia.
* Correspondng author: Bessem Samet
Email addresses: radens@beotel.rs; sradenovic@mas.bg.ac.rs (Stojan Radenović), bsamet@ksu.edu.sa (Bessem Samet)
ORCID iDs: https://orcid.org/0000-0001-8254-6688 (Stojan Radenović), https://orcid.org/0000-0002-6769-3417 (Bessem Samet)

In this case, the pair (M, D) is called a multiplicative metric space.

Let (M, D) be a multiplicative metric space and $\{u_n\}$ be a sequence in M.
We say that $\{u_n\}$ is multiplicative convergent to $u \in M$, if for all $\varepsilon > 1$, there exists a natural number n_0 such that $D(u_n, u) < \varepsilon$ for all $n \geq n_0$, or, equivalently,

$$\lim_{n\to\infty} D(u_n, u) = 1. \tag{1}$$

It can be easily seen that, if $\{u_n\}$ is a multiplicative convergent sequence, then there exists a unique $u \in M$ such that (1) holds.
We say that $\{u_n\}$ is a multiplicative Cauchy sequence, if for all $\varepsilon > 1$, there exists a natural number n_0 such that $D(u_n, u_m) < \varepsilon$ for all $n, m \geq n_0$, or, equivalently,

$$\lim_{n,m\to\infty} D(u_n, u_m) = 1.$$

If every multiplicative Cauchy sequence $\{u_n\} \subset M$ is multiplicative convergent to a certain $u \in M$, then (M, D) is called a complete multiplicative metric space.

The study of fixed point theory in multiplicative metric spaces was first considered in the paper [17] by Özavşar and Çevikel, where they extended three fixed point theorems: Banach's contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem, from (standard) metric spaces to multiplicative metric spaces. We recall below the standard versions of these three theorems.

Theorem 1.1 (Banach's contraction principle [4]). *Let (M, d) be a complete metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, 1)$ such that*

$$d(Fx, Fy) \leq \lambda d(x, y), \quad x, y \in M. \tag{2}$$

Then F possesses one and only one fixed point.

Theorem 1.2 (Kannan's fixed point theorem [15]). *Let (M, d) be a complete metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d(Fx, Fy) \leq \lambda \left[d(x, Fx) + d(y, Fy)\right], \quad x, y \in M.$$

Then F possesses one and only one fixed point.

Theorem 1.3 (Chatterjea's fixed point theorem [7]). *Let (M, d) be a complete metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d(Fx, Fy) \leq \lambda \left[d(x, Fy) + d(y, Fx)\right], \quad x, y \in M.$$

Then F possesses one and only one fixed point.

Özavşar and Çevikel [17] obtained the following multiplicative versions of the above fixed point theorems.

Theorem 1.4 (A multiplicative version of Banach's contraction principle [17]). *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, 1)$ such that*

$$D(Fx, Fy) \leq [D(x, y)]^{\lambda}, \quad x, y \in M. \tag{3}$$

Then F possesses one and only one fixed point.

Theorem 1.5 (A multiplicative version of Kannan's fixed point theorem [17]). *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$D(Fx, Fy) \leq [D(x, Fx)D(y, Fy)]^{\lambda}, \quad x, y \in M.$$

Then F possesses one and only one fixed point.

Theorem 1.6 (A multiplicative version of Chatterjea's fixed point theorem [17]). *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$D(Fx, Fy) \leq \lambda \left[D(x, Fy)D(y, Fx)\right]^{\lambda}, \quad x, y \in M.$$

Then F possesses one and only one fixed point.

Other fixed point results in multiplicative metric spaces can be found in [1, 3, 13, 14, 18, 19, 21–23] (see also the references therein).

Let D (resp. d) be a multiplicative metric (resp. metric) on M. We denote by d_D the mapping $d_D : M \times M \to [0, \infty)$ defined by

$$d_D(x, y) = \ln(D(x, y)), \quad x, y \in M.$$

We denote by D_d the mapping $D_d : M \times M \to [1, \infty)$ defined by

$$D_d(x, y) = e^{d(x,y)}, \quad x, y \in M.$$

Basing on the following observations:

(ι) If D is a multiplicative metric on M, then d_D is a metric on M;

($\iota\iota$) If (M, D) is a complete multiplicative metric space, then (M, d_D) is a complete metric space;

($\iota\iota\iota$) If d is a metric on M, then D_d is a multiplicative metric on M;

($\iota\iota\iota\iota$) If (M, d) is a complete metric space, then (M, D_d) is a complete multiplicative metric space,

Došenović et al. [9] (see also [2, 10, 11]) established that several fixed point results obtained in multiplicative metric spaces are equivalent to the corresponding fixed point results in metric spaces. In order to explain the used approach in [9], consider for instance a mapping $F : M \to M$ satisfying (3), where (M, D) is a complete multiplicative metric space. Applying the Logarithmic function in both sides of (3), we obtain

$$\ln(D(Fx, Fy)) \leq \lambda \ln(D(x, y)), \quad x, y \in M,$$

that is,

$$d_D(Fx, Fy) \leq \lambda d_D(x, y), \quad x, y \in M.$$

Then, by (ι), ($\iota\iota$) and Theorem 1.1, we obtain that F possesses one and only one fixed point. This shows that Theorem 1.1 implies Theorem 1.4. Conversely, consider a mapping $F : M \to M$ satisfying (2), where (M, d) is a complete metric space. Applying the exponential function in both sides of (2), we obtain

$$e^{d(Fx,Fy)} \leq \left[e^{d(x,y)}\right]^{\lambda}, \quad x, y \in M,$$

that is,

$$D_d(Fx, Fy) \leq [D_d(x, y)]^{\lambda}, \quad x, y \in M.$$

Then, by ($\iota\iota\iota$), ($\iota\iota\iota\iota$) and Theorem 1.4, we obtain that F possesses one and only one fixed point. This shows that Theorem 1.4 implies Theorem 1.1. Consequently, Theorems 1.1 and 1.4 are equivalent. Following the same approach, we can show that Theorems 1.2 and 1.5 (resp. Theorems 1.3 and 1.6) are equivalent.

Motivated by [9], it is natural to ask whether it is possible to find new contractions in multiplicative metric spaces, which cannot be reduced to contractions in metric spaces. A positive answer is given in this paper. Namely, we establish new extensions of Theorems 1.1 and 1.2 from metric spaces to multiplicative metric spaces, so that the used approach in [9] cannot be applied. The paper includes some examples showing the validity of our obtained results.

Our main results are provided in Section 2. Namely, in Subsection 2.1, we fix some notations that will be used throughout this paper. In Subsection 2.2, we obtain a new multiplicative version of Banach's contraction principle by considering the class of mappings $F : M \to M$ (where (M, D) is a multiplicative metric) satisfying contractions of the form

$$\varphi(D(Fx, Fy)) \leq \lambda \varphi(D(x, y))$$

for all $x, y \in M$ with $D(x, y) > 1$, where $\lambda \in (0, 1)$ is a constant and $\varphi : [1, \infty) \to [0, \infty)$ is a given function. Under certain conditions on φ, we show that F possesses one and only one fixed point. We also show that the used approach in [9] is inapplicable. In Subsection 2.3, a new multiplicative version of Kannan's contraction principle (see Theorem 1.2) is established by considering the class of mappings $F : M \to M$ (where (M, D) is a multiplicative metric) satisfying

$$\varphi(D(Fx, Fy)) \leq \lambda \left[\varphi(D(x, Fx)) + \varphi(D(y, Fy))\right]$$

for all $x, y \in M$ with $D(x, y) > 1$, where $\lambda \in (0, \frac{1}{2})$ is a constant and $\varphi : [1, \infty) \to [0, \infty)$ is a given function. Unlike Kannan's fixed point theorem in metric spaces, we show that under certain conditions, the mapping F may have more than one fixed point. An example is provided to illustrate this fact. Moreover, the uniqueness of the fixed point is obtained under additional conditions on φ.

2. Main results

We start this section by fixing some notations that will be used throughout this paper. We refer the reader to [6], where some suggestions are provided for unifying the terminology in fixed point theory.

2.1. Notations

We denote by M an arbitrary nonempty set. For a given mapping $F : M \to M$, we denote by $\mathrm{Fix}(F)$ the set of fixed points of F, that is,

$$\mathrm{Fix}(F) = \{u \in M : Fu = u\}.$$

For $u \in M$, the Picard sequence $\{F^n u\}$ is defined by

$$F^0 u = u, \quad F^{n+1} u = F(F^n u), \quad n \geq 0.$$

We denote by Φ the set of functions $\varphi : [1, \infty) \to [0, \infty)$ satisfying:

$$\varphi(s) \geq c(\ln s)^{\sigma}, \quad s > 1 \tag{4}$$

for some constants $c, \sigma > 0$.

Remark 2.1. *Observe that from* (4)*, we have*

$$\varphi(s) > 0, \quad s > 1.$$

We set

$$\Phi^{\uparrow} = \{\varphi \in \Phi : \varphi \text{ is nondecreasing}\}.$$

Let us provide some examples of functions $\varphi \in \Phi^{\uparrow}$.

Example 2.2. *Let*

$$\varphi(s) = a + bs^{\gamma} + cs^{\sigma}, \quad s \geq 1,$$

where $c, \sigma > 0$ and $a, b, \gamma \geq 0$. Then φ is nondecreasing and

$$\varphi(s) \geq cs^{\sigma} \geq c(\ln s)^{\sigma}, \quad s > 1.$$

Example 2.3. *Let*

$$\varphi(s) = \begin{cases} \frac{1}{2} & \text{if} \quad s = 1, \\ s & \text{if} \quad s > 1. \end{cases}$$

Then φ is nondecreasing. Moreover, for all $s > 1$, we have

$$\varphi(s) = s \geq \ln s.$$

Example 2.4. *Let $p \geq 1$ be a natural number and*

$$\varphi(s) = \sum_{i=1}^{p} c_i (\ln s)^{\sigma_i}, \quad s \geq 1,$$

where $c_i, \sigma_i \geq 0$ for all $i \in \{1, \cdots, p\}$ and $c_{i_0}\sigma_{i_0} \neq 0$ for some $i_0 \in \{1, \cdots, p\}$. Clearly, the function φ is nondecreasing. Moreover, for all $s > 1$, we have

$$\varphi(s) \geq c_{i_0} (\ln s)^{\sigma_{i_0}}.$$

Finally, we set

$$\Phi_0 = \{\varphi \in \Phi : \varphi(1) = 0\}.$$

2.2. A multiplicative version of Banach's contraction principle

In this subsection, we provide a new extension of Banach's contraction principle from metric spaces to multiplicative metric spaces. Namely, we have the following fixed point result.

Theorem 2.5. *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *There exist $\lambda \in (0, 1)$ and $\varphi \in \Phi$ such that*

$$\varphi(D(Fx, Fy)) \leq \lambda \varphi(D(x, y)) \tag{5}$$

for all $x, y \in M$ with $D(x, y) > 1$;

(ii) *For all $u, v \in M$, if*

$$\lim_{n \to \infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k} u\}$ of $\{F^n u\}$ such that

$$\lim_{k \to \infty} D(F^{n_k+1} u, Fv) = 1.$$

Then F possesses one and only one fixed point.

Proof. We first prove that, if the set of fixed points of F is nonempty, then it contains one and only one fixed point. Indeed, suppose that $u, v \in \text{Fix}(F)$ with $u \neq v$. Then, making use of (5) with $(x, y) = (u, v)$, we obtain

$$\varphi(D(Fu, Fv)) \leq \lambda \varphi(D(u, v)),$$

that is,

$$\varphi(D(u, v)) \leq \lambda \varphi(D(u, v)). \tag{6}$$

On the other hand, since $u \neq v$, then $D(u, v) > 1$, which implies (see Remark 2.1) that

$$\varphi(D(u, v)) > 0.$$

Then, dividing (6) by $\varphi(D(u,v))$, we reach a contradiction with $0<\lambda<1$.

We now establish that $\mathrm{Fix}(F)\neq\emptyset$. Indeed, for an arbitrary $x_0\in M$, let $\{x_n\}\subset M$ be the Picard sequence defined by

$$x_n = F^n x_0, \quad n\geq 0.$$

There are two possible cases.
Case 1: There exists $n_0\geq 0$ such that $x_{n_0}=x_{n_0+1}$.
In this case, obviously, x_{n_0} is the unique fixed point of F.
Case 2: $x_n\neq x_{n+1}$ for all $n\geq 0$.
In this case, making use of (5) with $(x,y)=(x_0,x_1)$, we obtain

$$\varphi(D(Fx_0,Fx_1))\leq \lambda\varphi(D(x_0,x_1)),$$

that is,

$$\varphi(D(x_1,x_2))\leq \lambda\varphi(D(x_0,x_1)). \tag{7}$$

Similarly, using (5) with $(x,y)=(x_1,x_2)$, we obtain

$$\varphi(D(x_2,x_3))\leq \lambda\varphi(D(x_1,x_2)),$$

which gives by (7) that

$$\varphi(D(x_2,x_3))\leq \lambda^2\varphi(D(x_0,x_1)).$$

Continuing in the same way, we get by induction that

$$\varphi(D(x_n,x_{n+1}))\leq \lambda^n\varphi(D(x_0,x_1)), \quad n\geq 0. \tag{8}$$

On the other hand, we obtain by (4) that

$$c\left[\ln\left(D(x_n,x_{n+1})\right)\right]^{\sigma}\leq \varphi(D(x_n,x_{n+1})), \quad n\geq 0. \tag{9}$$

Then, it follows from (8) and (9) that

$$\ln\left(D(x_n,x_{n+1})\right)\leq \left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}}\kappa^n, \quad n\geq 0, \tag{10}$$

where $\kappa=\lambda^{\frac{1}{\sigma}}\in(0,1)$. Next, making use of the multiplicative triangle inequality, for all $n\geq 0$ and $m\geq 1$, we obtain

$$D(x_n,x_{n+m})\leq D(x_n,x_{n+1})\cdots D(x_{n+m-1},x_{n+m}),$$

which implies by (10) that

$$\begin{aligned}\ln\left(D(x_n,x_{n+m})\right) &\leq \ln\left(D(x_n,x_{n+1})\right)+\cdots+\ln\left(D(x_{n+m-1},x_{n+m})\right)\\ &\leq \left(\kappa^n+\cdots+\kappa^{n+m-1}\right)\left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}}\\ &= \frac{\kappa^n(1-\kappa^m)}{1-\kappa}\left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}}\\ &\leq \frac{\kappa^n}{1-\kappa}\left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}},\end{aligned}$$

which is equivalent to

$$D(x_n,x_{n+m})\leq e^{\frac{\kappa^n}{1-\kappa}\left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}}}\to 1 \text{ as } n\to\infty.$$

Consequently, $\{x_n\}$ is a multiplicative Cauchy sequence in the complete multiplicative metric space (M, D). Then, there exists $\bar{x} \in M$ such that

$$\lim_{n\to\infty} D(x_n, \bar{x}) = 1, \tag{11}$$

which implies by (ii) that $\{x_n\}$ admits a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} D(x_{n_k+1}, F\bar{x}) = 1. \tag{12}$$

Hence, it follows from (11), (12) and the uniqueness of the limit that $\bar{x} \in \mathrm{Fix}(F)$. This completes the proof of Theorem 2.5. □

Remark 2.6. *We point out that, if* (5) *holds for all $x, y \in M$, then taking $x = y$, we get*

$$\varphi(1) \leq \lambda\varphi(1),$$

which yields (since $0 < \lambda < 1$) $\varphi(1) = 0$. So, in order to avoid this condition on φ, (5) *is assumed to be satisfied only for $x, y \in M$ with $D(x, y) > 1$ (or, equivalently, $x \neq y$).*

Remark 2.7. *Clearly, if F is a continuous mapping, then condition (ii) of Theorem 2.5 holds.*

Observe that the mapping $d_{\varphi(D)} : M \times M \to [0, \infty)$ defined by

$$d_{\varphi(D)}(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ \ln\left[\varphi(D(x, y))\right] & \text{if} \quad x \neq y \end{cases}$$

is not necessarily a metric on M. So, in the general case, (5) cannot be reduced to a contraction in a metric space. Therefore, the used approach in [9] cannot be applied in our case. The following example shows this fact.

Example 2.8. *Let $M = \{k_1, k_2, k_3\}$ and $D : M \times M \to [1, \infty)$ be the mapping defined by*

$$D(k_i, k_i) = 1,\ D(k_i, k_j) = D(k_j, k_i),\ D(k_1, k_2) = 2,\ D(k_1, k_3) = 3,\ D(k_2, k_3) = 4. \tag{13}$$

Remark that

$$\begin{aligned} D(k_1, k_3)D(k_3, k_2) &= 12 > 2 = D(k_1.k_2), \\ D(k_1, k_2)D(k_2, k_3) &= 8 > 3 = D(k_1, k_3), \\ D(k_2, k_1)D(k_1, k_3) &= 6 > 4 = D(k_2, k_3), \end{aligned}$$

which shows that D is a multiplicative metric on M.

Consider the function $\varphi : [1, \infty) \to [0, \infty)$ defined by

$$\varphi(s) = \begin{cases} \frac{3}{2}s & \textit{if} \quad 1 \leq s \leq 2, \\ \frac{5}{3}s & \textit{if} \quad 2 < s \leq 3, \\ \frac{1}{4}s & \textit{if} \quad s > 3. \end{cases}$$

Observe that for all $s > 1$, we have

$$\varphi(s) \geq \frac{1}{4}\ln s,$$

which shows that $\varphi \in \Phi$ (indeed, φ satisfies (4) *with $c = \frac{1}{4}$ and $\sigma = 1$).*

On the other hand, we have

$$\varphi(3) = 5 > 3 = \varphi(2)\varphi(4),$$

which implies that

$$\ln[\varphi(3)] > \ln[\varphi(2)] + \ln[\varphi(4)].$$

The above inequality is equivalent to

$$\ln[\varphi(D(k_1,k_3))] > \ln[\varphi(D(k_1,k_2))] + \ln[\varphi(D(k_2,k_3))],$$

that is,

$$d_{\varphi(D)}(k_1,k_3) > d_{\varphi(D)}(k_1,k_2) + d_{\varphi(D)}(k_2,k_3),$$

which shows that $d_{\varphi(D)}$ is not a metric on M.

We provide below an example to illustrate Theorem 2.5.

Example 2.9. *Let $M = \{k_1,k_2,k_3\}$ and $F : M \to M$ be the mapping defined by*

$$Fk_1 = k_1,\ Fk_2 = k_3,\ Fk_3 = k_1.$$

Let $D : M \times M \to [1,\infty)$ be the multiplicative metric defined by (13)*.*

Observe that there is no $\lambda \in (0,1)$ such that

$$D(Fk_1,Fk_2) \le [D(k_1,k_2)]^{\lambda}.$$

Indeed, if a such λ exists, then by the definition of F, we get

$$D(k_1,k_3) \le [D(k_1,k_2)]^{\lambda},$$

that is,

$$3 \le 2^{\lambda},$$

which is equivalent to

$$\lambda \ge \frac{\ln 3}{\ln 2} > 1.$$

Hence, we reach a contradiction with $\lambda \in (0,1)$. This shows that Theorem 1.4 due to Özavşar and Çevikel [17] is inapplicable in this example.

Consider now the function $\varphi : [1,\infty) \to [0,\infty)$ defined by

$$\varphi(s) = \begin{cases} 0 & \text{if} \quad s = 1, \\ 2s & \text{if} \quad 1 < s \le 2, \\ s & \text{if} \quad 2 < s \le 3, \\ 3s & \text{if} \quad s > 3. \end{cases}$$

Remark that for all $s > 1$, we have

$$\varphi(s) \ge \ln s,$$

which shows that $\varphi \in \Phi$ (indeed, φ satisfies (4) *with $c = \sigma = 1$). On the other hand, by the definition of the function φ, for $(i,j) = (1,2)$, we have*

$$\begin{aligned} \frac{\varphi(D(Fk_i,Fk_j))}{\varphi(D(k_i,k_j))} &= \frac{\varphi(D(Fk_1,Fk_2))}{\varphi(D(k_1,k_2))} \\ &= \frac{\varphi(D(k_1,k_3))}{\varphi(D(k_1,k_2))} \\ &= \frac{\varphi(3)}{\varphi(2)} \\ &= \frac{3}{4}. \end{aligned}$$

For $(i,j)=(1,3)$*, we have*

$$\begin{aligned}\varphi(D(Fk_i,Fk_j)) &= \varphi(D(Fk_1,Fk_3))\\ &= \varphi(D(k_1,k_1))\\ &= \varphi(1)\\ &= 0.\end{aligned}$$

For $(i,j)=(2,3)$*, we have*

$$\begin{aligned}\frac{\varphi(D(Fk_i,Fk_j))}{\varphi(D(k_i,k_j))} &= \frac{\varphi(D(Fk_2,Fk_3))}{\varphi(D(k_2,k_3))}\\ &= \frac{\varphi(D(k_3,k_1))}{\varphi(D(k_2,k_3))}\\ &= \frac{\varphi(3)}{\varphi(4)}\\ &= \frac{1}{4}.\end{aligned}$$

Consequently, F satisfies (5) *for all* $\frac{3}{4}\leq\lambda<1$*, which shows that condition (i) of Theorem 2.5 is satisfied.*

Moreover, for all $u\in M$*, we have by the definition of F that*

$$F^n u = k_1, \quad n\geq 2.$$

So, if $\lim_{n\to\infty} D(F^n u,v)=1$*, then* $v=k_1$ *and*

$$\lim_{n\to\infty} D(F^{n+1}u,Fv) = D(Fk_1,Fk_1) = 1,$$

which shows that condition (ii) of Theorem 2.5 is also satisfied. Thus, Theorem 2.5 applies. On the other hand, k_1 *is the unique fixed point of F, which confirms our obtained result.*

We now study the particular case, when $\varphi\in\Phi^{\uparrow}$. We have the following result.

Proposition 2.10. *Let* (M,D) *be a multiplicative metric space and* $F: M\to M$ *be a given mapping. If F satisfies condition (i) of Theorem 2.5 with* $\varphi\in\Phi^{\uparrow}$*, then condition (ii) of Theorem 2.5 holds.*

Proof. Let $u,v\in M$ be such that

$$\lim_{n\to\infty} D(F^n u,v)=1. \tag{14}$$

If there exists some $n_0\geq 0$ such that $F^n u = v$ for all $n\geq n_0$, then

$$\lim_{n\to\infty} D(F^{n+1}u,Fv) = D(Fv,Fv) = 1.$$

Otherwise, there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$F^{n_k}u \neq v, \quad k\geq 0,$$

that is,

$$D(F^{n_k}u,v) > 1, \quad k\geq 0.$$

Then, making use of (5) with $(x,y)=(F^{n_k}u,v)$, we obtain

$$\varphi(D(F^{n_k+1}u,Fv)) \leq \lambda\varphi(D(F^{n_k}u,v)) \leq \varphi(D(F^{n_k}u,v)), \quad k\geq 0,$$

which implies (since φ is nondecreasing) that

$$1\leq D(F^{n_k+1}u,Fv)\leq D(F^{n_k}u,v), \quad k\geq 0.$$

Then, making use of (14) and passing to the limit as $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = 1.$$

This shows that condition (ii) of Theorem 2.5 is satisfied. □

From Theorem 2.5 and Proposition 2.10, we deduce immediately the following result.

Corollary 2.11. *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that there exist $\lambda \in (0, 1)$ and $\varphi \in \Phi^{\uparrow}$ such that (5) holds. Then F possesses one and only one fixed point.*

2.3. A multiplicative version of Kannan's fixed point theorem

In this subsection, we provide a new extension of Kannan's contraction principle (see Theorem 1.2) from metric spaces to multiplicative metric spaces. Namely, we have the following fixed point result.

Theorem 2.12. *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *There exist $\lambda \in (0, \frac{1}{2})$ and $\varphi \in \Phi$ such that*

$$\varphi(D(Fx, Fy)) \le \lambda \left[\varphi(D(x, Fx)) + \varphi(D(y, Fy))\right] \tag{15}$$

for all $x, y \in M$ with $D(x, y) > 1$;

(ii) *For all $u, v \in M$, if*

$$\lim_{n\to\infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = 1.$$

Then F possesses at least one fixed point.

Proof. For an arbitrary $x_0 \in M$, let $\{x_n\} \subset M$ be the Picard sequence defined by

$$x_n = F^n x_0, \quad n \ge 0.$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \ge 0$, then x_{n_0} is a fixed point of F. Assume now that $x_n \ne x_{n+1}$ for all $n \ge 0$, that is,

$$D(x_n, x_{n+1}) > 1, \quad n \ge 0.$$

Then, using (15) with $(x, y) = (x_0, x_1)$, we obtain

$$\varphi(D(Fx_0, Fx_1)) \le \lambda \left[\varphi(D(x_0, Fx_0)) + \varphi(D(x_1, Fx_1))\right],$$

that is,

$$\varphi(D(x_1, x_2)) \le \lambda \left[\varphi(D(x_0, x_1)) + \varphi(D(x_1, x_2))\right],$$

which yields

$$\varphi(D(x_1, x_2)) \le \kappa\varphi(D(x_0, x_1)), \tag{16}$$

where $\kappa = \frac{\lambda}{1-\lambda} \in (0, 1)$. Similarly, using (15) with $(x, y) = (x_1, x_2)$, we obtain

$$\varphi(D(x_2, x_3)) \le \lambda \left[\varphi(D(x_1, x_2)) + \varphi(D(x_2, x_3))\right],$$

which implies that

$$\varphi(D(x_2, x_3)) \leq \kappa\varphi(D(x_1, x_2)). \tag{17}$$

Then, from (16) and (17), we get

$$\varphi(D(x_2, x_3)) \leq \kappa^2\varphi(D(x_0, x_1)).$$

Continuing this process, we get by induction that

$$\varphi(D(x_n, x_{n+1})) \leq \kappa^n\varphi(D(x_0, x_1)), \quad n \geq 0.$$

So, by (4), we get

$$\ln(D(x_n, x_{n+1})) \leq \left[\frac{\varphi(D(x_0, x_1))}{c}\right]^{\frac{1}{\sigma}} \alpha^n, \quad n \geq 0,$$

where $\alpha = \kappa^{\frac{1}{\sigma}} \in (0, 1)$. Proceeding as in the proof of Theorem 2.5, for all $n \geq 0$ and $m \geq 1$, we obtain

$$D(x_n, x_{n+m}) \leq e^{\frac{\alpha^n}{1-\alpha}\left[\frac{\varphi(D(x_0,x_1))}{c}\right]^{\frac{1}{\sigma}}} \to 1 \text{ as } n \to \infty,$$

which shows that $\{x_n\}$ is a multiplicative Cauchy sequence in the complete multiplicative metric space (M, D). Then, there exists $\bar{x} \in M$ such that

$$\lim_{n\to\infty} D(x_n, \bar{x}) = 1, \tag{18}$$

which implies by (ii) that $\{x_n\}$ admits a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} D(x_{n_k+1}, F\bar{x}) = 1. \tag{19}$$

Finally, (18) together with (19) implies that $\bar{x}$ is a fixed point of F. This completes the proof of Theorem 2.12. □

Remark 2.13. *Notice that, if* (15) *holds for all $x, y \in M$, then taking $x = y$, we get*

$$\varphi(1) \leq 2\lambda\varphi(D(x, Fx)), \quad x \in M.$$

In particular, if $x = Fx$, then

$$\varphi(1) \leq 2\lambda\varphi(1),$$

which implies (since $0 < \lambda < \frac{1}{2}$) that $\varphi(1) = 0$. In order to avoid this condition on φ, (15) *is supposed to be satisfied only for $x, y \in M$ with $D(x, y) > 1$.*

Unlike the Kannan fixed point theorem in metric spaces (see Theorem 1.2), the following example shows that under the conditions of Theorem 2.12, the mapping F may admit more than one fixed point.

Example 2.14. *Let $M = \{k_1, k_2, k_3\}$ and $F : M \to M$ be the mapping defined by*

$$Fk_1 = k_1, \ Fk_2 = k_2, \ Fk_3 = k_2.$$

Let $D : M \times M \to [1, \infty)$ be the multiplicative metric defined by (13).

We introduce the function $\varphi : [1, \infty) \to [0, \infty)$ defined by

$$\varphi(s) = \begin{cases} 3 & \text{if} \quad s = 1, \\ \frac{1}{2}(s+2) & \text{if} \quad 1 < s < 4, \\ \frac{7}{4}s & \text{if} \quad s \geq 4. \end{cases}$$

Observe that

$$\varphi(s) \geq \frac{1}{2}\ln s, \quad s > 1,$$

which shows that $\varphi \in \Phi$ *(indeed,* φ *satisfies* (4) *with* $c = \frac{1}{2}$ *and* $\sigma = 1$*).*

Let us now calculate

$$R(i,j) := \frac{\varphi(D(Fk_i, Fk_j))}{\varphi(D(k_i, Fk_i)) + \varphi(D(k_j, Fk_j))}, \quad (i,j) \in \{(1,2),(1,3),(2,3)\}.$$

For $(i,j) = (1,2)$*, we get*

$$\begin{aligned} R(1,2) &= \frac{\varphi(D(Fk_1, Fk_2))}{\varphi(D(k_1, Fk_1)) + \varphi(D(k_2, Fk_2))} \\ &= \frac{\varphi(D(k_1, k_2))}{\varphi(D(k_1, k_1)) + \varphi(D(k_2, k_2))} \\ &= \frac{\varphi(2)}{2\varphi(1)} \\ &= \frac{1}{3}. \end{aligned}$$

For $(i,j) = (1,3)$*, we get*

$$\begin{aligned} R(1,3) &= \frac{\varphi(D(Fk_1, Fk_3))}{\varphi(D(k_1, Fk_1)) + \varphi(D(k_3, Fk_3))} \\ &= \frac{\varphi(D(k_1, k_2))}{\varphi(D(k_1, k_1)) + \varphi(D(k_3, k_2))} \\ &= \frac{\varphi(2)}{\varphi(1) + \varphi(4)} \\ &= \frac{1}{5}. \end{aligned}$$

Finally, for $(i,j) = (2,3)$*, we obtain*

$$\begin{aligned} R(2,3) &= \frac{\varphi(D(Fk_2, Fk_3))}{\varphi(D(k_2, Fk_2)) + \varphi(D(k_3, Fk_3))} \\ &= \frac{\varphi(D(k_2, k_2))}{\varphi(D(k_2, k_2)) + \varphi(D(k_3, k_2))} \\ &= \frac{\varphi(1)}{\varphi(1) + \varphi(4)} \\ &= \frac{3}{10}. \end{aligned}$$

From the above calculations, we deduce that

$$R(i,j) \leq \frac{1}{3}, \quad (i,j) \in \{(1,2),(1,3),(2,3)\},$$

which shows that (15) *is satisfied for all* $\frac{1}{3} \leq \lambda < \frac{1}{2}$*.*

We now show that condition (ii) of Theorem 2.12 is satisfied. Remark first that for all $i \in \{1,2,3\}$*, there exists* $j \in \{1,2,3\}$ *such that*

$$F^n k_i = k_j, \quad n \geq 2. \tag{20}$$

Indeed, by the definition of F, we have

$$F^n k_1 = k_1, \quad n \geq 2,$$
$$F^n k_2 = k_2, \quad n \geq 2,$$
$$F^n k_3 = k_2, \quad n \geq 2.$$

Hence, if for some $i \in \{1, 2, 3\}$, *we have*

$$\lim_{n\to\infty} D(F^n k_i, q) = 1,$$

then by (20), *we get* $q = k_j$ *and*

$$\lim_{n\to\infty} D(F^{n+1}k_i, Fq) = D(Fk_j, Fq) = D(Fq, Fq) = 1.$$

Consequently, condition (ii) of Theorem 2.12 holds. Then, Theorem 2.12 applies. Remark that in this example, we have

$$\text{Fix}(F) = \{k_1, k_2\}.$$

We now show that, if $\varphi \in \Phi^{\uparrow}$ in Theorem 2.12, then F possesses one and only one fixed point.

Theorem 2.15. *Assume that all the conditions of Theorem 2.12 hold, where* $\varphi \in \Phi^{\uparrow}$. *Then F possesses one and only one fixed point.*

Proof. From Theorem 2.12, we know that the set of fixed points of F is nonempty. So, we have just to prove that F admits a unique fixed point. We argue by contradiction supposing that $\bar{x}$ and $\bar{y}$ are two distinct fixed points of F, that is,

$$D(\bar{x}, \bar{y}) > 1, \;\; \bar{x} = F\bar{x}, \;\; \bar{y} = F\bar{y}.$$

Then, using (15) with $(x, y) = (\bar{x}, \bar{y})$, we obtain

$$\varphi(D(F\bar{x}, F\bar{y})) \leq \lambda \left[\varphi(D(\bar{x}, F\bar{x})) + \varphi(D(\bar{y}, F\bar{y}))\right],$$

that is,

$$\varphi(D(\bar{x}, \bar{y})) \leq 2\lambda\varphi(1).$$

Since $2\lambda < 1$, it holds that

$$\varphi(D(\bar{x}, \bar{y})) \leq \varphi(1).$$

Next, using that φ is nondecreasing, we obtain

$$D(\bar{x}, \bar{y}) = 1.$$

Hence, we reach a contradiction with $D(\bar{x}, \bar{y}) > 1$. This completes the proof of Theorem 2.15. □

We now consider the case, when $\varphi \in \Phi_0$.

Theorem 2.16. *Assume that all the conditions of Theorem 2.12 hold, where* $\varphi \in \Phi_0$. *Then F possesses one and only one fixed point.*

Proof. From Theorem 2.12, we know that the set of fixed points of F is nonempty. So, we have just to prove that F admits a unique fixed point. From the proof of Theorem 2.15, if $\bar{x}$ and $\bar{y}$ are two distinct fixed points of F, then

$$0 \leq \varphi(D(\bar{x}, \bar{y})) \leq \varphi(1) = 0,$$

which implies that

$$\varphi(D(\bar{x}, \bar{y})) = 0. \tag{21}$$

On the other hand, from Remark 2.1, we know that $\varphi(s) > 0$ for all $s > 1$. Then, from (21), we deduce that

$$D(\bar{x}, \bar{y}) = 1,$$

and we reach a contradiction with $\bar{x} \neq \bar{y}$. This completes the proof of Theorem 2.16. □

3. Conclusion

Due to the numerous applications of multiplicative calculus (see e.g. [5]), several works have been devoted to the development of fixed point theory in multiplicative metric spaces. Unfortunately, it was shown that the most fixed point results obtained in multiplicative metric spaces are equivalent to the corresponding fixed point results in metric spaces (see [9]). So, in order to get more significant fixed point results in multiplicative metric spaces, it is natural to ask whether it is possible to find new contractions in multiplicative metric spaces, which cannot be reduced to contractions in metric spaces. This question is investigated in this paper. Namely, making use of a function $\varphi : [1,\infty) \to [0,\infty)$ satisfying

$$\varphi(s) \geq c(\ln s)^{\sigma}, \quad s > 1,$$

where $c, \sigma > 0$ are constants, we considered contractions on a multiplicative metric space (M, D) involving the mapping $\varphi(D)$, where

$$\varphi(D)(x,y) = \varphi(D(x,y)), \quad x, y \in M.$$

In this way, the mapping $d_{\varphi(D)} : M \times M \to [0,\infty)$ defined by

$$d_{\varphi(D)}(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ \ln\left[\varphi(D(x,y))\right] & \text{if} \quad x \neq y, \end{cases}$$

is not necessarily a metric on M (see Example 2.8). So, the used approach in [9] consisting in reducing a contraction on a multiplicative metric space to a contraction on a metric space, cannot be used.

Following the above idea, a new multiplicative version of Banach's contraction principle is established (see Theorem 2.5). A further result is also obtained in the special case, when the function φ is nondecreasing (see Corollary 2.11). Next, a new multiplicative version of Kannan's fixed point theorem is obtained (see Theorem 2.12). It is interesting to mention that unlike Kannan's fixed point theorem in metric spaces, under the assumptions of Theorem 2.12, the mapping $F : M \to M$ may have more than one fixed point (see Example 2.14). Further results are also obtained in the special cases, when φ is nondecreasing (see Theorem 2.15) or $\varphi(1) = 0$ (see Theorem 2.16). Namely, in Theorems 2.15 and 2.16, the uniqueness of the fixed point is proved.

In this work, only multiplicative versions of the Banach and Kannan fixed point theorems are provided. It would be interesting to extend the present study to other fixed point theorems, for instance, Chatterjea's fixed point theorem [7], Reich's fixed point theorem [20], Ćirić's fixed point theorem [8], Meir-Keeler's fixed point theorem [16], etc.

References

[1] M. Abbas, B. Ali, Y.I. Suleiman, *Common fixed points of locally contractive mappings in multiplicative metric spaces with application*, Int. J. Math. Math. Sci. **2015** (2015), 1–7.

[2] R.P. Agarwal, E. Karapinar, B. Samet, *An essential remark on fixed point results on multiplicative metric spaces*, Fixed Point theory Appl. **2016** (2016), 1–3.

[3] MU. Ali, S. Sessa, Y. Almalki, M. Alansari, *Fundamental characteristics of the product-operated metric spaces*, Axioms. **13** (2024), 1–13.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundam. Math. **3** (1922), 133–181.

[5] A.E. Bashirov, E.M. Kurpınar, A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (2008) 36–48.

[6] V. Berinde, A. Petruşel, Ioan A. Rus, *Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces*, Fixed Point Theory. **24** (2023), 525–540.

[7] S.K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[8] L. Ćirić, *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc. **45** (1974), 267–273.

[9] T. Došenović, M. Postolache, S. Radenović, *On multiplicative metric spaces: survey*, Fixed Point Theory Appl. **2016** (2016), 1–17.

[10] T. Došenović, S. Radenović, *Some critical remarks on the paper: An essential remark on fixed point results on multiplicative metric spaces*, J. Adv. Math. Stud. **10** (2017), 20–24.

[11] T. Došenović, S. Radenović, *Multiplicative metric spaces and contractions of rational type*, Adv. Theor. Nonlin. Anal. Appl. **2** (2018), 195–201.

[12] M. Grossman, R. Katz, *Non-Newtonian Calculus*, Lee Press, Pigeon Cove, 1972.

[13] X. He, M. Song, D. Chen, *Common fixed points for weak commutative mappings on a multiplicative metric space*, Fixed Point Theory Appl. **2014** (2014), 1–9.

[14] Hizbullah, S.U. Rehman, S.U. Khan, G. Rahman, K. Nonlaopon, *A new approach to integral equations via contraction results in multiplicative metric spaces*, AIMS Mathematics. **7** (2022), 19891–19901.

[15] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.

[16] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

[17] M. Özavşar, A.C. Çevikel, *Fixed point of multiplicative contraction mappings on multiplicative metric spaces*, Journal of Engineering Technology and Applied Sciences. **2** (2017), 65–79.

[18] Y. Piao, *Unique fixed points for four non-continuous mappings satisfying* Ψ*-contractive condition on non-complete multiplicative metric spaces*, Adv. Fixed Point Theory. **9** (2019), 135–145.

[19] T. Rasham, M.S. Shabbir, P. Agarwal, S. Momani, *On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications*, Fuzzy Sets Syst. **437** (2022), 81–96.

[20] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124.

[21] B. Rome, M. Sarwar, *Characterization of multiplicative metric completeness*, Int. J. Anal. Appl. **10** (2016), 90–94.

[22] V. Srinivas, T. Tirupathi, K. Mallaiah, *A fixed point theorem using EA property on multiplicative metric space*, J. Math. Comput. Sci. **10** (2020), 1788–1800.

[23] O. Yamaod, W. Sintunavarat, *Some fixed point results for generalized contraction mappings with cyclic* (α, β)*-admissible mapping in multiplicative metric spaces*, J. Inequal. Appl. **2014** (2014), 1–15.