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Uniqueness of solution to stochastic scalar conservation laws on bounded domain

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Abstract. This paper is concerned with the uniqueness of stochastic entropy solutions for stochastic scalar conservation law forced by a multiplicative noise on a bounded domain with a non-homogeneous boundary condition. The uniqueness is obtained by using the method of Kruzhkov's doubling variables.

1. Introduction

Let D be a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂D. Let $T > 0$ be arbitrarily fixed. Set $Q = (0,T)\times D$ and $\Sigma = (0,T)\times\partial D$. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ be a given probability set-up. In this paper, we are interested in the first order stochastic conservation laws driven by a multiplicative noise of the following type

$$du - div(f(u))dt = h(u)dw(t), \quad \text{in } \Omega\times Q, \tag{1}$$

with initial condition

$$u(0,\cdot) = u_0(\cdot), \quad \text{in } D, \tag{2}$$

and boundary condition

$$u = a, \quad \text{on } \Sigma, \tag{3}$$

for a random scalar-valued function $u : (\omega, t, x) \in \Omega \times [0,T] \times D \mapsto u(\omega, t, x) =: u(t, x) \in \mathbb{R}$, where $f = (f_1, ..., f_N) : \mathbb{R} \to \mathbb{R}^N$ is a differentiable vector field standing for the flux, $h : \mathbb{R} \to \mathbb{R}$ is measurable and $w = \{w(t)\}_{0\le t\le T}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$. The initial data $u_0 : D \subset \mathbb{R}^N \to \mathbb{R}$ will be specified later and we suppose that the boundary data $a : \Sigma \to \mathbb{R}$ is measurable.

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Problem (1)-(3) was studied recently by Kobayasi and Noboriguchi in [18] via kinetic solution approach. By introducing a notion of kinetic formulations in which the kinetic defect measures on the boundary of domain are turncated, they obtained the well-posedness of (1)-(3).

When $h = 0$, the deterministic problem (1)-(3) is well studied by many authors in the literature, see for example [1, 29]. The authors of [29] studied the problem (1)-(3) with $h = 0$ in L^1-setting. In order to deal with unbounded solutions, they have defined a notion of renormalized entropy solution which generalizes the definition of entropy solutions introduced by Otto in [28] in the L^∞ frame work. They have proved existence and uniqueness of such generalized solution in the case when f is locally Lipschitz and the boundary data a verifies the following condition: $f_{max}(a) \in L^1(\Sigma)$, where f_{max} is the "maximal effective flux" defined by

$$f_{max}(a) = \{\sup |f(u)|, \quad u \in [-a^-, a^+]\}.$$

They gave an example to illustrate that the assumption $a \in L^1(\Sigma)$ is not enough in order to prove a priori estimates in $L^1(Q)$, and that the assumption should be $f_{max}(a) \in L^1(\Sigma)$. Furthermore, in [1], the authors revisited the problem (1)-(3) and introduced a notion of entropy solution to the problem (1)-(3) with $h = 0$. Following [1], an entropy solution of (1)-(3) is a function $u \in L^\infty(Q)$ satisfying

$$\begin{aligned} -\int_\Sigma \xi\Upsilon^+(x,k,a(t,x)) &\le \int_Q [(u-k)^+\xi_t - \chi_{u>k}(f(u)-f(k))\cdot\nabla\xi] \\ &\quad + \int_D (u_0-k)^+\xi(0,\cdot) \quad \text{and} \end{aligned} \tag{4}$$

$$\begin{aligned} -\int_\Sigma \xi\Upsilon^-(x,k,a(t,x)) &\le \int_Q [(k-u)^-\xi_t - \chi_{k>u}(f(k)-f(u))\cdot\nabla\xi] \\ &\quad + \int_D (k-u_0)^-\xi(0,\cdot) \end{aligned} \tag{5}$$

for any $\xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, $\xi \ge 0$ and for all $k \in \mathbb{R}$, where

$$\Upsilon^+(x,k,a) := \max_{k\le r,s\le a\vee k} |(f(r)-f(s))\cdot\vec{n}(x)|, \quad \Upsilon^-(x,k,a) := \max_{a\wedge k\le r,s\le k} |(f(r)-f(s))\cdot\vec{n}(x)|$$

for any $k \in \mathbb{R}$, a.e. $x \in \partial D$, and $\vec{n}$ denoting the unit outer normal to ∂D. Here and in what follows, $a \wedge k := \min\{a,k\}$ and $a \vee k := \max\{a,k\}$. It is remarked that the above definition of entropy solution is a natural extension of the definition of that given by Otto [28].

Having a stochastic forcing term $h(u)dw(t)$ in Equation (1) is very natural for problem modeling arising in a wide variety of fields in physics, engineering, biology, jut mention a few. The Cauchy problem of equation (1) with additive noise has been studied in [17] wherein Kim proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution. Moreover, a Kruzhkov-type method was used there to prove the uniqueness. In [30], Vallet and Wittbold extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. By utilising the vanishing viscosity method, Young measure techniques and Kruzhkov doubling variables technique, they managed to show the existence and uniqueness of the stochastic entropy solution.

On the other other, concerning the case of multiplicative noise, for Cauchy problem over the whole spatial space, Feng and Nualart in [12] introduced a notion of strong entropy solution in order to prove the uniqueness for the entropy solution. Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solution only in $1D$ case. Chen et al. [5] considered high space dimensional problem and they proved that the multi-dimensional stochastic problem is well-posedness by using a uniform spatial BV-bound. Bauzet et al.[2] proved a result of existence and uniqueness of the weak measure-valued entropy solution to the multi-dimensional Cauchy problem.

Using a kinetic formulation, Debussche and Vovelle [7] obtained a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus, (also see [16, 18]). Konatar [20] obtained the uniqueness for stochastic scalar conservation laws on Riemannian manifolds revisited by using the kinetic formulation and doubling of variables. Lv et al. [25] considered the Kinetic solutions for nonlocal stochastic conservation laws, also see [27].

Just recently, Bauzet et al. [3] studied the problem (1)-(3) with $a = 0$ (i.e., the homogeneous boundary condition). Under the assumptions that the flux function f and h satisfy the global Lipschitz condition, they obtained the existence and uniqueness of measure-valued solution to problem (1)-(3) with $a = 0$. Lv et al. [24] extended the result of [3] to the stochastic nonlocal conservation law. Gess-Souganidis [13] considered the scalar conservation laws with multiple rough fluxes.

In paper [26], the authors have established the existence and uniqueness of stochastic entropy solutions to the initial boundary value problem (1)-(3). But the proof of uniqueness is too long. The reason is that the boundary value is considered in the proof, that is to say, we used the following fact $(r-s)^+ = (r\vee k - s\vee k)^+ + (r\wedge k - s\wedge k)^+$ for any $r, s, k \in \mathbb{R}$, where k is the maximum value of the boundary function. And $u \vee k$ will be approximated by $k + \eta_\delta(u-k)$. Therefore, it will be complicated. The aim of this paper is to shorten the proof of uniqueness. Meanwhile, it is worthing noting that, unlike that in [3], the definition of stochastic entropy solutions needs the information of boundary value. More precisely, when $\varphi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$, the constant k must be positive in $\mu_{\eta,k}(\varphi) \geq 0$, see Definition 2.1 in [3]. However, in the next section, definition 2.1 shows that $k \in \mathbb{R}$ if $\varphi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$. For the whole space, Drivas, et al [11] considered the invariant measures for stochastic conservation laws on the line. The small time asymptotic, ergodicity and central limit theorem and moderate deviation principle of scalar stochastic conservation laws are obtained by [9], [10] and [31], respectively.

The paper is organized as follows. In section 2, we introduce the notion of stochastic entropy solution for (1)-(3) and state the main results. Section 3 is devoted to the proof of uniqueness. We end up this section with introducing some notations.

Notations. In general, if $G \subset \mathbb{R}^N$, $\mathcal{D}(G)$ denotes the restriction of functions $u \in \mathcal{D}(\mathbb{R}^N)$ to G such that $support(u) \cap G$ is compact. The notation $\mathcal{D}^+(G)$ stands for the subset of non-negative elements of $\mathcal{D}(G)$.

For a given separable Banach space X, we denote by $N_w^2(0,T,X)$ the space of the predictable X-valued processes. This space is the space $L^2((0,T)\times\Omega, X)$ for the product measure $dt\otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (i.e. the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $(s,t)\times A$ for any $A \in \mathcal{F}_s$, for $t > s > 0$).

Denote $\mathcal{E}^+$ the totality of non-negative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \to x^+$ such that $\eta(x) = 0$ if $x \leq 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then η'' has a compact support and η and η' are Lipschitz-continuous functions. $\mathcal{E}^-$ denotes the set $\{\breve{\eta} := \eta(-\cdot), \eta \in \mathcal{E}^+\}$ and $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$. Then, for convenience, denote

$$sgn_0^+(x) = 1 \text{ if } x > 0 \text{ and } 0 \text{ else}; \quad sgn_0^-(x) = -sgn_0^+(-x) \quad sgn_0 = sgn_0^+ + sgn_0^-,$$
$$F(a,b) = sgn_0(a-b)[f(a)-f(b)]; \quad F^{+(-)}(a,b) = sgn_0^{+(-)}(a-b)[f(a)-f(b)],$$
$$\text{and for any } \eta \in \mathcal{E}, \quad F^\eta(a,b) = \int_b^a \eta'(\sigma - b) f'(\sigma)d\sigma.$$

2. Entropy solution and Main result

The aim of this section is to give a definition of entropy solution and to state the main result. Following the idea of [2], we have the definitions 2.1 and 2.2.

For convenience, for any function u of $N_w^2(0,T;L^2(D))$, any real number k and any regular function $\eta \in \mathcal{E}^+$, denote dP-a.s. in Ω by $\mu_{\eta,k}$, the distribution in D defined by

$$\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) \quad := \quad & \int_D \eta(u_0-k)\varphi(0)dx + \int_Q \left(\eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi\right)dxdt \\
& + \int_Q \eta'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\varphi dxdt \\
& + \int_\Sigma \eta'(a-k)\varphi\Upsilon^+(x,k,a(t,x))dSdt; \\
\varphi \mapsto \mu_{\breve{\eta},k}(\varphi) \quad := \quad & \int_D \breve{\eta}(u_0-k)\varphi(0)dx + \int_Q \left(\breve{\eta}(u-k)\partial_t\varphi - F^{\breve{\eta}}(u,k)\nabla\varphi\right)dxdt
\end{aligned}$$

$$+\int_Q \check{\eta}'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \check{\eta}''(u-k)h^2(u)\varphi dxdt$$
$$+\int_\Sigma \check{\eta}'(a-k)\varphi \Upsilon^-(x,k,a(t,x))dSdt,$$

where $\Upsilon^+(x,k,a(t,x))$ and $\Upsilon^-(x,k,a(t,x))$ are defined as in Introduction.

Definition 2.1. *A function u of $N^2_w(0,T;L^2(D))$ is an entropy solution of stochastic conservation law (1) with the initial condition $u_0 \in L^p(D)$ and boundary condition $a \in L^\infty(\Sigma)$, if $u \in L^2(0,T;L^2(\Omega;L^p(D)))$, $p = 2,3,\cdots$ and*

$$\mu_{\eta,k}(\varphi) \geq 0, \qquad \mu_{\check{\eta},k}(\varphi) \geq 0 \quad dP - a.s.,$$

where $\varphi \in \mathcal{D}^+((0,T\times\mathbb{R}^N))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}^+$ and $\check{\eta} \in \mathcal{E}^-$.

For technical reasons, we need to consider a generalized notion of entropy solution. In fact, in the first step, we will only prove the existence of a Young measure-valued solution, see [2, Appendix A.3] for the basic knowledge of Young measures. Then, thanks to a result of uniqueness, we will be able to deduce the existence of an entropy solution in the sense of Definition 2.1.

Definition 2.2. *A function u of $N^2_w(0,T;L^2(D\times(0,1))) \cap L^\infty(0,T;L^p(\Omega\times D\times(0,1)))$ is a Young measure-valued solution of stochastic conservation law (1) with the initial condition $u_0 \in L^p(D)$ and boundary condition $a \in L^\infty(\Sigma)$, $p = 2,3,\cdots$, if*

$$\int_0^1 \mu_{\eta,k}(\varphi)d\alpha \geq 0, \qquad \int_0^1 \mu_{\check{\eta},k}(\varphi)d\alpha \geq 0 \quad dP - a.s.,$$

where $\varphi \in \mathcal{D}^+((0,T\times\mathbb{R}^N))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}^+$, $\alpha \in (0,1)$ and $\check{\eta} \in \mathcal{E}^-$.

Throughout this paper, we assume that

(H_1): The flux function $f : \mathbb{R} \mapsto \mathbb{R}^N$ is of class C^2, its derivatives have at most polynomial growth, $f(0) = 0_{\mathbb{R}^N}$, and f'' is bounded in $\mathbb{R}$ if $a \not\equiv 0$;

(H_2): $h : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$;

(H_3): $u_0 \in L^p(D)$, $p \geq 2$ and $a \in C(\Sigma)$.

The main result of this paper is:

Theorem 2.3. *Under assumptions $H_1 - H_3$ there exists a unique measure-valued entropy solution in sense of Definition 2.2 and this solution is obtained by viscous approximation.*

It is an unique entropy solution in sense of Definition 2.1.

If u_1, u_2 are entropy solutions of (1) corresponding to initial data u_{01}, $u_{02} \in L^p(D)$ and the boundary data a_1, $a_2 \in L^\infty(\Sigma)$, respectively, then for any $t \in (0,T)$

$$\mathbb{E}\int_D |u_1 - u_2| \leq \int_D |u_{01} - u_{02}|dx + \int_\Sigma \max_{min(a_1,a_2)\leq r,s\leq max(a_1,a_2)} |(f(r) - f(s))\cdot \vec{n}(x)|.$$

3. Uniqueness

The aim of this section is to prove

Theorem 3.1. *The solution given by Theorem 2.3 is the unique measure-valued entropy solution in the sense of Definition 2.2.*

Let us consider the following stochastic parabolic problem

$$\begin{cases} du^\varepsilon - [\varepsilon\Delta u^\varepsilon + div(f(u^\varepsilon))]dt = h(u^\varepsilon)dw(t) & \text{in } Q, \\ u_\varepsilon(0,x) = u_0^\varepsilon(x) & \text{in } D, \\ u^\varepsilon = a^\varepsilon & \text{on } \Sigma, \end{cases} \tag{1}$$

where we assume that $a^\varepsilon \in C^\infty(\Sigma)$, $\|a^\varepsilon\|_{C^1} \le \|a\|_{L^\infty}$ and $a^\varepsilon \to a$ in $L^1(\Sigma)$. Moreover, a^ε is the trace on Σ of a function $U \in C([0,T]\times\bar{D})$ such that $\partial_t U \in C^{\gamma,0}([0,T]\times D)$, $\Delta U \in C^{\gamma,0}([0,T]\times D)$, $U(t,\cdot) \in W^{2,p}(D)$ for some $\gamma \in (0,1)$ and for any $p > 1$.

The following comparison result plays a crucial role in proof of Theorem 3.1.

Lemma 3.2. *Let u_1, u_2 be Young measure-valued entropy solutions to (1) with initial data $u_{01}, u_{02} \in L^2(D)$, $\alpha, \beta \in (0,1)$ and boundary data $a_1, a_2 \in L^\infty(\Sigma)$, respectively, and assume that at least one of them is obtained by viscous approximation. Then for any $\varphi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$*

$$\begin{aligned} -\int_\Sigma \varphi\Upsilon^+(x,a_2,a_1) \le\ & \mathbb{E}\int_Q\int_0^1\int_0^1 (u_1(t,x,\alpha)-u_2(t,x,\beta))^+\partial_t\varphi d\alpha d\beta dx dt \\ & -\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_1(t,x,\alpha),u_2(t,x,\beta))\cdot\nabla\varphi d\alpha d\beta dx dt \\ & +\int_D (u_{01}-u_{02})^+\varphi(0)dx. \end{aligned}$$

Proof. As usual we use Kruzhkov's technique of doubling variables [21, 22] in order to prove the comparison result. We choose two pairs of variables (t,x) and (s,y) and consider u_1 as a function of $(t,x) \in Q$ and u_2 as a function of $(s,y) \in Q$. For any $r > 0$, let $\{B_i^r\}_{i=0,\cdots,m_r}$ be a covering of $\bar{D}$ satisfying $B_0^r \cap \partial D = \emptyset$, and such that, for each $i \ge 1$, B_i^r is a ball of diameter $\le r$, contained in some larger ball $\tilde{B}_i^r$ with $\tilde{B}_i^r \cap \partial D$ is part of the graph of a Lipschitz function. Let $\{\phi_i^r\|_{i=0,\cdots,m_r}$ denote a partition of unity subordinate to the covering $\{B_i^r\}_i$. Let $\varphi \in \mathcal{D}^+((0,T)\times\mathbb{R}^N)$.

Note that, due to the fact that both functions u_1, u_2 satisfy the classical stochastic semi-Kruzhkov entropy inequalities (neglecting the boundary effect) for any $k \in \mathbb{R}$ in $\mathcal{D}'((0,T)\times D)$, one can prove exactly as in [3] that u_1, u_2 satisfy the following local comparison principle: for any $\xi \in \mathcal{D}^+(Q)$,

$$\begin{aligned} 0 \le\ & \mathbb{E}\int_Q\int_0^1\int_0^1 (u_1(t,x,\alpha)-u_2(t,x,\beta))^+\partial_t\xi d\alpha d\beta dx dt \\ & -\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_1(t,x,\alpha),u_2(t,x,\beta))\cdot\nabla\xi d\alpha d\beta dx dt \\ & +\int_D (u_{01}-u_{02})^+\xi(0)dx. \end{aligned} \tag{2}$$

In particular, (2) holds with $\xi = \varphi\phi_0^r$. Now, let $i \in \{1,\cdots,m_r\}$ be fixed in the following. For simplicity, we omit the dependence on r and i and simply set $\phi = \phi_i^r$ and $B = B_i^r$. We choose a sequence of mollifiers $(\rho_n)_n$ in $\mathbb{R}^N$ such that $x \mapsto \rho_n(x-y) \in \mathcal{D}$ for all $y \in B$. $\sigma_n(x) = \int_D \rho_n(x-y)dy$ is an increasing sequence for all $x \in B$ and $\sigma_n(x) = 1$ for all $x \in B$ with $dist(x,\mathbb{R}^N\setminus D) > \frac{c}{n}$ for some $c = c(i,r)$ depending on $B = B_i^r$. Let $(\varrho_m)_m$ denote a sequence of mollifiers in $\mathbb{R}$ with $supp\varrho_m \subset (-\frac{2}{m}, 0)$.

Define the test function

$$\zeta_{m,n}(t,x,s,y) = \varphi(s,y)\phi(y)\rho_n(y-x)\varrho_m(t-s)$$

Note that, for m, n sufficiently large

$$(t,x) \mapsto \zeta_{m,n}(t,x,s,y) \in \mathcal{D}((0,T)\times\mathbb{R}^N), \quad \text{for any } (s,y) \in Q,$$

$$(s,y)\mapsto \zeta_{m,n}(t,x,s,y)\in \mathcal{D}(Q), \qquad\qquad \text{for any } (t,x)\in Q.$$

Let $u_2^\varepsilon(s,y)$ be the solution of (1) with initial data u_{02}^ε and boundary data a_2^ε, and $\eta_\delta\in\mathcal{E}^+$ satisfying $\eta_\delta(\cdot)\mapsto(\cdot)^+$ and $\eta'_\delta(\cdot)\mapsto sgn_0^+(\cdot)$ as $\delta\to 0$. Then taking $\zeta_{m,n}(t,x,s,y)$ as test function in Definition 2.2, for a. e. $(t,x)\in Q$, we have

$$\begin{aligned}
0 \;\le\; & \int_0^1\int_Q \left[\eta_\delta(u_1-k)(\zeta_{m,n})_t - F^{\eta_\delta}(u_1,k)\cdot\nabla_x\zeta_{m,n}\right]dxdtd\alpha\\
&+\int_0^1\int_Q \eta'_\delta(u_1-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\\
&+\frac{1}{2}\int_0^1\int_Q h^2(u_1)\eta''_\delta(u_1-k)\zeta_{m,n}\\
&+\int_\Sigma \eta'_\delta(a_1-k)\zeta_{m,n}\Upsilon^+(x,k,a_1)\\
&+\int_D \eta'_\delta(u_{01}-k)\zeta_{m,n}(0,x,s,y)dx.
\end{aligned}$$

Multiplying the above inequality by $\varrho_l(k-u_2^\varepsilon)$ and integrating in k and (t,x) over $\mathbb{R}$ and Q, respectively, and taking expectation, we have

$$\begin{aligned}
0 \;\le\; & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D \eta_\delta(u_{01}-k)\zeta_{m,n}(0,x,s,y)dx\varrho_l(k-u_2^\varepsilon)dkdyds\\
&+\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u_1-k)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds\\
&-\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 F^{\eta_\delta}(u_1,k)\varphi\phi\varrho_m\cdot\nabla_x\rho_n(y-x)d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds\\
&+\frac{1}{2}\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 h^2(u_1)\eta''_\delta(u_1-k)\zeta_{m,n}d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds\\
&+\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta'_\delta(u_1-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k-u_2^\varepsilon)dkdyds\\
&+\mathbb{E}\int_Q\int_{\mathbb{R}}\int_\Sigma \eta'_\delta(a_1-k)\zeta_{m,n}\Upsilon^+(x,k,a_1)dSdt\varrho_l(k-u_2^\varepsilon)dkdyds\\
:=\;& I_1+I_2+\cdots+I_6.
\end{aligned}$$

As u_2^ε is a viscous solution, the Itô formula applied to $\int_D\eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}dy$ yields that for a.e. $(t,x)\in Q$

$$\begin{aligned}
0 \;\le\; & \int_D \eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}(t,x,0,y)dy+\int_Q \eta_\delta(k-u_2^\varepsilon)(\zeta_{m,n})_sdyds\\
&-\varepsilon\int_Q \eta'_\delta(k-u_2^\varepsilon)\Delta u_2^\varepsilon\zeta_{m,n}dyds-\int_Q F^{\check{\eta}_\delta}(k,u_2^\varepsilon)\cdot\nabla_y\zeta_{m,n}dyds\\
&+\frac{1}{2}\int_Q \eta''_\delta(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds-\int_Q \eta'_\delta(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s),
\end{aligned}$$

where we used the fact that for any fixed $(t,x)\in Q$, $\zeta_{m,n}(t,x,s,y)\in\mathcal{D}(Q)$ and

$$\begin{aligned}
\int_D d\left[\eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}\right]dy \;=\; & \int_D \eta_\delta(k-u_2^\varepsilon)\left(\zeta_{m,n}\right)_s dsdy+\frac{1}{2}\int_D \eta''_\delta(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dsdy\\
&-\int_D \eta'_\delta(k-u_2^\varepsilon)\zeta_{m,n}\left[\left(\varepsilon\Delta u^\varepsilon+div(f(u^\varepsilon))\right)dt-h(u^\varepsilon)dw(t)\right]dy.
\end{aligned}$$

Multiplying the above inequality by $\varrho_l(u_1-k)$ and integrating in k over $\mathbb{R}$, in (t,x) over Q and in α over $(0,1)$,respectively, and taking expectation, we have

$$
\begin{aligned}
0 \quad \leq \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}(t,x,0,y)\varrho_l(u_1-k)d\alpha dkdydxdt \\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)(\partial_s\varphi\varrho_m+\varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
& -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-u_2^\varepsilon)\Delta_y u_2^\varepsilon\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\tilde{\eta}_\delta}(u_2^\varepsilon,k)\cdot\nabla_y\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
& +\frac{1}{2}\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta''_\delta(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s)\varrho_l(u_1-k)d\alpha dkdxdt \\
:= \quad & J_1+J_2+\cdots+J_6.
\end{aligned}
$$

Noting that $\varrho_m(t)=0$, $t\in[0,T]$, we have

$$
\begin{aligned}
I_1+J_1 \quad = \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u_1-k)\zeta_{m,n}(0,x,s,y)\varrho_l(k-u_2^\varepsilon)d\alpha dkdydxds \\
= \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u_1-k)\varphi\phi\rho_n\varrho_m(-s)\varrho_l(k-u_2^\varepsilon)d\alpha dkdydxds \\
\longrightarrow_{m,n,\varepsilon,l,\delta} \quad & \mathbb{E}\int_D(u_{01}-u_{02})^+\varphi(0,x)\phi(x)dx.
\end{aligned}
$$

Due to $u_1\in N_w^2(0,T,L^2(D))$, u_{01}, $u_{02}\in L^2(D)$ and the compact support of $\zeta_{m,n}$, we know that the convergences in above inequality hold, see [2] for the similar proof.

By using the fact $\partial_t\varrho_m(t-s)+\partial_s\varrho_m(t-s)=0$ and changing variable technique, we get

$$
\begin{aligned}
I_2+J_2 \quad = \quad & \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u_1-k)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds \\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)(\partial_s\varphi\varrho_m+\varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
= \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
& +\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u_1-u_2^\varepsilon-\tau)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(\tau)d\tau dxdtdyds \\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(u_1-u_2^\varepsilon-\tau)\varphi\phi\rho_n\partial_s\varrho_m(t-s)dyds\varrho_l(\tau)d\alpha d\tau dxdt \\
= \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
\rightarrow_{l,\delta,m,\varepsilon} \quad & \mathbb{E}\int_Q\int_0^1\int_0^1 (u_1(t,x,\alpha)-u_2(t,x,\beta))^+\partial_t\varphi(t,x)\phi(x)d\alpha d\beta dxdt.
\end{aligned}
$$

By using again the fact that for any fixed $(t,x) \in Q$, $\zeta_{m,n}(t,x,s,y) \in \mathcal{D}(Q)$ and Hölder inequality, we obtain

$$\begin{aligned}
J_3 &= -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta'(k-u_2^\varepsilon)\Delta_y u_2^\varepsilon \zeta_{m,n} dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&= \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \left(\Delta_y\eta_\delta(k-u_2^\varepsilon) - \eta_\delta'(k-u_2^\varepsilon)|\nabla u_2^\varepsilon|^2\right)\zeta_{m,n} dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\leq \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \Delta_y\eta_\delta(k-u_2^\varepsilon)\zeta_{m,n} dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&= \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)\Delta_y\zeta_{m,n} dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\to_{l,\delta} \varepsilon\mathbb{E}\int_Q\int_Q\int_0^1 (u_1-u_2^\varepsilon)^+\Delta_y\zeta_{m,n} dydsdxdt \\
&\leq \varepsilon\mathbb{E}\int_Q\int_Q\int_0^1 |u_1|\Delta_y\zeta_{m,n} dydsdxdt + \varepsilon\mathbb{E}\sup_{0\leq t\leq T}\|u_2^\varepsilon\|_{L^2(D)} \\
&\quad \times\int_Q\int_0^T\int_0^1\left[\int_D\left(\Delta_y(\varphi(s,y)\phi(y)\rho_n(y-x))\right)^2dy\right]^{\frac{1}{2}}\varrho_m d\alpha dsdxdt \\
&\to_\varepsilon 0,
\end{aligned}$$

where we used $\mathbb{E}\sup_{0\leq t\leq T}\|u_2^\varepsilon\|_{L^2(D)}$ is uniformly bounded for $\varepsilon>0$.

Noting that $\nabla_x\rho_m(y-x)+\nabla_y\rho_m(y-x)=0$, we have

$$\begin{aligned}
I_3+J_4 &= -\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 F^{\eta_\delta}(u_1,k)\varphi\phi\varrho_m\cdot\nabla_x\rho_n(y-x)d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds \\
&\quad -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\check{\eta}_\delta}(u_2^\varepsilon,k)\cdot(\rho_n\nabla_y(\varphi\phi)+\varphi\phi\nabla_y\rho_n)\varrho_m dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\to_l -\mathbb{E}\int_Q\int_Q\int_0^1 F^{\eta_\delta}(u_1,u_2^\varepsilon)\varphi\phi\varrho_m\cdot\nabla_x\rho_n(y-x)d\alpha dxdtdyds \\
&\quad -\mathbb{E}\int_Q\int_Q\int_0^1 F^{\eta_\delta}(u_1,u_2^\varepsilon)\varphi\phi\varrho_m\cdot\nabla_y\rho_n(y-x)d\alpha dxdtdyds \\
&\quad -\mathbb{E}\int_Q\int_Q\int_0^1 F^{\eta_\delta}(u_1,u_2^\varepsilon)\cdot\rho_n\nabla_y(\varphi\phi)\varrho_m dydsd\alpha dxdt \\
&\to_{m,\delta,\varepsilon,n} -\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_1(t,x,\alpha),u_2(t,x,\beta))\nabla(\varphi(t,x)\phi(x))d\alpha d\beta dxdt.
\end{aligned}$$

$$\begin{aligned}
I_4+J_5 &= \frac{1}{2}\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 h^2(u_1)\eta_\delta''(u_1-k)\zeta_{m,n}d\alpha\varrho_l(k-u_2^\varepsilon)dkdxdtdyds \\
&\quad +\frac{1}{2}\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta''(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\to_{l,m} \frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta''(u_1-u_2^\varepsilon)\left(h^2(u_1)+h^2(u_2^\varepsilon)\right)\varphi(t,y)\phi(y)\rho_n(y-x)dyd\alpha dxdt.
\end{aligned}$$

Now, we come to the estimate of most interesting part, the stochastic integrals. Since $\vartheta(t)=\int_0^1\varrho_l(u_1(t,x,\tau)-$

$k)d\tau$ is predictable and if one denotes

$$\iota(s) = \int_D \eta'_\delta(k - u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dy,$$

we have that

$$\mathbb{E}\left[\vartheta(t)\int_t^T \iota(s)dw(s)\right] = \mathbb{E}\left[\vartheta(t)\int_0^T \iota(s)dw(s)\right] - \mathbb{E}\left[\vartheta(t)\int_0^t \iota(s)dw(s)\right] = 0$$

because that

$$\mathbb{E}\left[\vartheta(t)\int_0^T \iota(s)dw(s)\right] = \mathbb{E}\left[\vartheta(t)\mathbb{E}\left(\int_0^T \iota(s)dw(s)|\mathcal{F}_t\right)\right] = \mathbb{E}\left[\vartheta(t)\int_0^t \iota(s)dw(s)\right].$$

Similarly, let $\vartheta\left(s - \frac{2}{m}\right) = \varrho_l(k - u_2^\varepsilon\left(s - \frac{2}{m}, y\right))$ and

$$\iota(t) = \int_D \int_0^1 \eta'_\delta(u_1 - k)h(u_1)\zeta_{m,n}dxd\alpha,$$

then we get that

$$\mathbb{E}\int_Q\int_{\mathbb{R}} \vartheta\left(s - \frac{2}{m}\right)\int_0^T \iota(t)dw(t)dkdyds = \int_Q\int_{\mathbb{R}} \mathbb{E}\vartheta\left(s - \frac{2}{m}\right)\int_{\left(s-\frac{2}{m}\right)^+}^s \iota(t)dw(t)dkdyds = 0.$$

Here for the first equality the compact property of ϱ_l is used and for the second equality the property of conditional expectation is used. Thus, we have

$$\begin{aligned} I_5 + J_6 &= \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta'_\delta(u_1 - k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k - u_2^\varepsilon)dkdyds \\ &\quad -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_t^T\int_0^1 \eta'_\delta(k - u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s)\varrho_l(u_1 - k)d\alpha dkdxdt \\ &= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\int_0^1 \eta'_\delta(u_1 - k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k - u_2^\varepsilon)dkdyds \\ &= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\int_0^1 \eta'_\delta(u_1 - k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha \\ &\quad \times\left[\varrho_l(k - u_2^\varepsilon(s, y)) - \varrho_l\left(k - u_2^\varepsilon\left(s - \frac{2}{m}, y\right)\right)\right]dkdyds \end{aligned}$$

As $du_2^\varepsilon = [\varepsilon\Delta u_2^\varepsilon + div(f(u_2^\varepsilon))]dt + h(u_2^\varepsilon)dw(t) := A_\varepsilon dt + h(u_2^\varepsilon)dw(t)$, by Itô formula, we arrive that

$$\begin{aligned} &\varrho_l(k - u_2^\varepsilon(s, y)) - \varrho_l\left(k - u_2^\varepsilon\left(s - \frac{2}{m}, y\right)\right) \\ = &-\int_{\left(s-\frac{2}{m}\right)^+}^s \varrho'_l(k - u_2^\varepsilon(\sigma, y))A_\varepsilon(\sigma, y)d\sigma \\ &-\int_{\left(s-\frac{2}{m}\right)^+}^s \varrho'_l(k - u_2^\varepsilon(\sigma, y))h(u_2^\varepsilon(\sigma, y))dw(\sigma) \\ &+\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^s \varrho''_l(k - u_2^\varepsilon(\sigma, y))h^2(u_2^\varepsilon(\sigma, y))d\sigma \\ = &-\frac{\partial}{\partial k}\left\{\int_{\left(s-\frac{2}{m}\right)^+}^s \varrho_l(k - u_2^\varepsilon(\sigma, y))A_\varepsilon(\sigma, y)d\sigma\right. \end{aligned}$$

$$+\int_{\left(s-\frac{2}{m}\right)^+}^{s}\varrho_l(k-u_2^\varepsilon(\sigma,y))h(u_2^\varepsilon(\sigma,y))dw(\sigma)$$
$$\left.-\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^{s}\varrho_l'(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma\right\}$$

Therefore,

$$\begin{aligned}
I_5+J_6 &= -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_D\int_0^1\eta_\delta''(u_1-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\\
&\quad\times\{\cdots\}dkdyds\\
&:= L_1+L_2+L_3.
\end{aligned}$$

Let us evaluate the limits of L_1, L_2 and L_3. Following [15], we know that the solution u_2^ε of (1) will belong to $L^p(D)$ if $u_{02}^\varepsilon\in L^p(D)$. We assume that $u_{02}^\varepsilon\in C^\infty(D)$ and u_{02}^ε converges to u_{02} in $L^2(D)$. Thus the solution $u_2^\varepsilon\in L^p(D)$, $\forall p\geq 2$. By using the properties of the heat kernel, one can prove that $u_2^\varepsilon\in W^{2,p}(D)$, see [6, 8, 33]. That is, $A_\varepsilon\in L^2(D)$. The proof of this part is similar to that of [3]. We first consider L_1:

$$\begin{aligned}
|L_1| &\leq \int_Q\int_{\mathbb{R}}\int_D\left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_0^1\eta_\delta''(u_1-k)h(u_1)d\alpha\zeta_{m,n}dw(t)\right)^2\right]^{\frac{1}{2}}\\
&\quad\times\left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^{s}\varrho_l(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\right)^2\right]^{\frac{1}{2}}dkdxdyds\\
&\leq \int_Q\int_{\mathbb{R}}\int_D\rho_n(y-x)\varphi(s,y)\phi(y)\left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_0^1\eta_\delta''(u_1-k)\varrho_m(t-s)h(u_1)\right)^2d\alpha dt\right]^{\frac{1}{2}}\\
&\quad\times\frac{2}{\sqrt{m}}\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}\left(\varrho_l(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)\right)^2d\sigma\right]^{\frac{1}{2}}dkdxdyds\\
&\leq Cl\sqrt{m}\int_Q\int_{\mathbb{R}}\int_D\rho_n(y-x)\left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_0^1\eta_\delta''(u_1-k)h(u_1)\right)^2d\alpha dt\right]^{\frac{1}{2}}\\
&\quad\times\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}1_{\{-\frac{2}{l}\leq k-u_2^\varepsilon(\sigma,y)\leq 0\}}A_\varepsilon^2(\sigma,y)d\sigma\right]^{\frac{1}{2}}dkdxdyds\\
&\leq Cl\sqrt{m}\int_Q\int_{\mathbb{R}}\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_0^1\left(\eta_\delta''(u_1-k)h(u_1)\right)^2d\alpha dtdkdxds\\
&\quad+Cl\sqrt{m}\int_Q\int_{\mathbb{R}}\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}1_{\{-\frac{2}{l}\leq k-u_2^\varepsilon(\sigma,y)\leq 0\}}A_\varepsilon^2(\sigma,y)d\sigma dkdyds\\
&\leq \frac{Cl\sqrt{m}}{\delta^2}\int_Q\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}\int_0^1\int_{\mathbb{R}}1_{\{u_1-\delta\leq k\leq u_1\}}h^2(u_1)dkd\alpha dtdxds\\
&\quad+Cl\sqrt{m}\int_Q\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^{s}A_\varepsilon^2(\sigma,y)d\sigma dyds\\
&\leq \frac{Cl}{\delta\sqrt{m}}\mathbb{E}\int_Q\int_0^1h^2(u_1)d\alpha dtdx+\frac{C}{\sqrt{m}}\mathbb{E}\int_QA_\varepsilon^2(s,y)dsdy\\
&\to_m 0.
\end{aligned}$$

Similarly, we get

$$
\begin{aligned}
|L_3| \quad \le \quad & \frac{1}{2}\Big|\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D\int_0^1 \eta''_\delta(u_1-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha \\
& \times\int_{(s-\frac{2}{m})^+}^{s} \varrho'_l(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma dkdyds\Big| \\
\le \quad & \frac{1}{2}\int_Q\int_D\int_{\mathbb{R}}\left[\mathbb{E}\left(\int_{(s-\frac{2}{m})^+}^{s}\int_0^1 \eta''_\delta(u_1-k)h(u_1)d\alpha\zeta_{m,n}dw(t)\right)^2\right]^{\frac{1}{2}} \\
& \times\left[\mathbb{E}\left(\int_{(s-\frac{2}{m})^+}^{s} \varrho'_l(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma\right)^2\right]^{\frac{1}{2}} dkdxdyds \\
\le \quad & \int_Q\int_D\int_{\mathbb{R}}\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s}\int_0^1 \left(\eta''_\delta(u_1-k)h(u_1)\zeta_{m,n}\right)^2 d\alpha dt\right]^{\frac{1}{2}} \\
& \times\frac{C}{\sqrt{m}}\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s} \left(\varrho'_l(k-u_2^\varepsilon(\sigma,y))\right)^2 h^4(u_2^\varepsilon(\sigma,y))d\sigma\right]^{\frac{1}{2}} dkdxdyds \\
\le \quad & C\sqrt{m}\int_Q\int_D\int_{\mathbb{R}}\rho_n(y-x)\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s}\int_0^1 \frac{1}{\delta^2}1_{\{u_1-\delta\le k\le u_1\}}h^2(u_1)d\alpha dt\right]^{\frac{1}{2}} \\
& \times\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s} l^2h^4(u_2^\varepsilon(\sigma,y))d\sigma\right]^{\frac{1}{2}} dkdxdyds \\
\le \quad & \frac{Cl}{\sqrt{m}}\left(\mathbb{E}\int_Q\int_0^1 h^2(u_1)d\alpha dtdx+\mathbb{E}\int_Q h^4(u_2^\varepsilon(s,y))dsdy\right) \\
\to_m \quad & 0,
\end{aligned}
$$

where we used the facts that $u_2^\varepsilon \in L^4(D)$ and $u_1 \in L^2(D)$. Thanks to Fubini's theorem and the properties of Itô integral, we have

$$
\begin{aligned}
\lim_m L_1+L_2+L_3 \quad = \quad & -\lim_m \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D\int_0^1 \eta''_\delta(u_1-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha \\
& \times\int_{(s-\frac{2}{m})^+}^{s} \varrho_l(k-u_2^\varepsilon(\sigma,y))h(u_2^\varepsilon(\sigma,y))dw(\sigma)dkdyds \\
= \quad & -\lim_m \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D\int_0^1 \eta''_\delta(u_1-k)h(u_1)\zeta_{m,n}d\alpha \\
& \times\varrho_l(k-u_2^\varepsilon(t,y))h(u_2^\varepsilon(t,y))dtdxdkdyds \\
\to_l \quad & -\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u_1-u_2^\varepsilon(t,y))h(u_1)\varphi(t,y)\phi(x)\rho_n(y-x)h(u_2^\varepsilon(t,y))d\alpha dtdxdy
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\lim_{m,l} I_4+J_5+I_5+J_6 \quad = \quad & \frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u_1-u_2^\varepsilon) \\
& \times\left(h^2(u_1)-2h(u_1)h(u_2^\varepsilon)+h^2(u_2^\varepsilon)\right)\varphi(t,y)\phi(y)\rho_n(y-x)dyd\alpha dxdt \\
= \quad & \frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u_1-u_2^\varepsilon)
\end{aligned}
$$

$$\times\Big(h(u_1)-h(u_2^\varepsilon)\Big)^2\varphi(t,y)\phi(y)\rho_n(y-x)dyd\alpha dxdt$$
$$\to_\delta \quad 0,$$

and thus

$$\lim_\delta\lim_{m,l} I_4+J_5+I_5+J_6\le 0.$$

Lastly, we consider I_6. By the assumptions of a_2^ε, we have

$$\begin{aligned} I_6 &= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_\Sigma \eta_\delta'(a_1-k)\zeta_{m,n}\Upsilon^+(x,k,a_1)dSdt\varrho_l(k-u_2^\varepsilon)dkdyds \\ &\to_{m,n,l,\varepsilon,\delta} \int_\Sigma \varphi\phi\Upsilon^+(x,a_2,a_1)dSdt. \end{aligned}$$

Combining all estimates yield

$$\begin{aligned} 0 \le\ & \mathbb{E}\int_D (u_{01}-u_{02})^+\varphi(0,x)\phi(x)dx \\ &+\mathbb{E}\int_Q\int_0^1\int_0^1 (u_1(t,x,\alpha)-u_2(t,x,\beta))^+\partial_t\varphi(t,x)\phi(x)d\alpha d\beta dxdt \\ &-\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_1(t,x,\alpha),u_2(t,x,\beta))\nabla(\varphi(t,x)\phi(x))d\alpha d\beta dxdt \\ &+\int_\Sigma \varphi\phi\Upsilon^+(x,a_2,a_1)dSdt. \end{aligned}$$

Summing over $i=0,1,\cdots,m_r$, taking into account the local inequality for $i=0$, we find, for any $\xi\in\mathcal{D}((0,T)\times\mathbb{R}^N)$,

$$\begin{aligned} 0 \le\ & \mathbb{E}\int_D (u_{01}-u_{02})^+\xi(0,x)dx \\ &+\mathbb{E}\int_Q\int_0^1\int_0^1 (u_1(t,x,\alpha)-u_2(t,x,\beta))^+\partial_t\xi(t,x)d\alpha d\beta dxdt \\ &-\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_1(t,x,\alpha),u_2(t,x,\beta))\nabla\xi(t,x)d\alpha d\beta dxdt \\ &+\int_\Sigma \xi\Upsilon^+(x,a_2,a_1)dSdt. \end{aligned}$$

Proof of Theorem 3.1 Now, we consider the second half. Similarly, as u_1 is a entropy solution, using the other half of Definition 2.2, and applying the Itô formula to $\int_D\eta_\delta(u_2^\varepsilon-k)$, we have

$$\begin{aligned} 0 \le\ & \mathbb{E}\int_D (u_{02}-u_{01})^+\xi(0,x)dx \\ &+\mathbb{E}\int_Q\int_0^1\int_0^1 (u_2(t,x,\beta)-u_1(t,x,\alpha))^+\partial_t\xi(t,x)d\alpha d\beta dxdt \\ &-\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u_2(t,x,\beta),u_1(t,x,\alpha))\nabla\xi(t,x)d\alpha d\beta dxdt \\ &+\int_\Sigma \xi\Upsilon^-(x,a_2,a_1)dSdt. \end{aligned}$$

Summing the above two inequalities, and using the fact $|a-b| = (a-b)^+ + (b-a)^+$, we have

$$\begin{aligned} 0 \leq\ & \mathbb{E}\int_D |u_{01}-u_{02}|\xi(0,x)dx \\ & +\mathbb{E}\int_Q\int_0^1\int_0^1 |u_1(t,x,\alpha)-u_2(t,x,\beta)|\partial_t\xi(t,x)d\alpha d\beta dxdt \\ & -\mathbb{E}\int_Q\int_0^1\int_0^1 F(u_1(t,x,\alpha),u_2(t,x,\beta))\nabla\xi(t,x)d\alpha d\beta dxdt \\ & +\int_\Sigma \xi \max_{\min(a_1,a_2)\leq r,s\leq \max(a_1,a_2)} |(f(r)-f(s))\cdot\vec{n}(x)|dSdt, \end{aligned} \tag{3}$$

where we used the fact

$$\Upsilon^-(x,a_2,a_1)+\Upsilon^+(x,a_2,a_1) = \max_{\min(a_1,a_2)\leq r,s\leq \max(a_1,a_2)} |(f(r)-f(s))\cdot\vec{n}(x)|.$$

Now, we prove the inequality in Theorem 2.3. For each $n\in\mathbb{N}$, define

$$\phi_n(x) = \begin{cases} 1 & \text{if } |x|\leq n, \\ 2(1-\frac{|x|}{2n}) & \text{if } n<|x|\leq 2n, \\ 0 & \text{if } |x|>2n. \end{cases}$$

For each $h>0$ and $0\leq t<T$, define

$$\psi_h(s) = \begin{cases} 1 & \text{if } s\leq t, \\ 1-\frac{s-t}{h} & \text{if } t<s\leq t+h, \\ 0 & \text{if } s>t+h. \end{cases}$$

Then, by standard approximation, truncation and mollification argument, (3) holds with

$$\xi(t,x) = \psi_h(s)\phi_n(x).$$

Define

$$A(s) = \mathbb{E}\left[\int_D\int_0^1\int_0^1 |u_1(s,x)-u_2(s,x)|dx\right],$$

then $A\in L^1_{loc}(0,T)$. It is easy to check that any right Lebesgue point of $A(s)$ is also a right Lebesgue point of

$$A_n(s) = \mathbb{E}\left[\int_D\int_0^1\int_0^1 |u_1(s,x)-u_2(s,x)|\phi_n(\cdot)(x)dx\right]$$

for all n. Let t be a right Lebesgue point of A. We choose this t in the definition of $\psi_h(s)$. Thus, (3) implies that

$$\begin{aligned} & \frac{1}{h}\int_t^{t+h}\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u_1(s,x)-u_2(s,x)|\phi_n(x)dx\right]ds \\ \leq\ & \mathbb{E}\int_Q\int_0^1\int_0^1 F(u_1(s,x),u_2(s,x))\nabla\phi_n(x)\psi_h(s)dxds \\ & +\mathbb{E}\left[\int_D |u_{01}(x)-u_{02}(x)|\phi_n(x)dx\right] \\ & +\int_\Sigma \psi_h(s)\phi_n(x)\max_{\min(a_1,a_2)\leq r,s\leq \max(a_1,a_2)} |(f(r)-f(s))\cdot\vec{n}(x)|dSdt. \end{aligned}$$

Taking limit as $h \to 0$, we know that there exists $p \geq 2$ such that

$$\begin{aligned}
&\mathbb{E}\left[\int_D \int_0^1 \int_0^1 |u_1(t,x) - u_2(t,x)|\phi_n(x)dx\right] \\
\leq\ & \mathbb{E}\int_Q \int_0^1 \int_0^1 F(u_1(s,x), u_2(s,x))\nabla\phi_n(x)dxds \\
&+\mathbb{E}\left[\int_D |u_{01}(x) - u_{02}(x)|\phi_n(x)dx\right] \\
&+\int_\Sigma \phi_n(x) \max_{\min(a_1,a_2)\leq r,s\leq \max(a_1,a_2)} |(f(r) - f(s))\cdot \vec{n}(x)|dSdt \\
\leq\ & \frac{C(T)}{n}\left[1 + \sup_{0\leq t\leq T} \mathbb{E}\|u_1(t)\|^p_{L^p(D)} + \sup_{0\leq t\leq T} \mathbb{E}\|u_2(t)\|^p_{L^p(D)}\right] \\
&+\mathbb{E}\left[\int_{\mathbb{R}^d} |u_{01}(x) - u_{02}(x)|dx\right] \\
&+\int_\Sigma \max_{\min(a_1,a_2)\leq r,s\leq \max(a_1,a_2)} |(f(r) - f(s))\cdot \vec{n}(x)|dSdt. \qquad (4)
\end{aligned}$$

Therefore, letting $n \to \infty$, we obtain the desired inequality. □

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References

[1] K. Ammar, P. Wittbold and J. Carrillo, Scalar conservation laws with general boundary condition and continuous flux function, J. Differential Equations **228** (2006) 111-139.
[2] C. Bauzet, G. Vallet and P. Wittbold, The Cauchy problem for conservation laws with a multiplicative stochastic perturbation, J. Hyperbolic Differential Equations **9** (2012) 661-709.
[3] C. Bauzet, G. Vallet and P. Wittbold, The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation, J. Funct. Anal. **266** (2014) 2503-2545.
[4] Ch. Castaing, P. Raynaud de Fitte and M. Valadier, Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory, Math. Apple., vol. 571, Kluwer Academic Publishers, Dordrecht, 2004.
[5] G. Q. Chen, Q. Ding and K. H. Karlsen, On nonlinear stochastic balance laws, Arch. Ration. Mech. Anal. **204** (2012) 707-743.
[6] Pao-Liu Chow, Stochastic partial differential equations, Chapman Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman Hall/CRC, Boca Raton, FL, 2007. x+281 pp. ISBN: 978-1-58488-443-9.
[7] A. Debussche and J. Vovelle, Scalar conservation law with stochastic forcing, J. Funct. Anal. **259** (2010) 1014-1042.
[8] A. Debussche, S. de Moor and M. Hofmanova, A regularity result for quasilinear stochastic partial differential equations of parabolic type, SIAM J. Math. Anal. **47** (2015) 1590-1614.
[9] Z. Dong, R. Zhang, On the small time asymptotics of scalar stochastic conservation laws, Appl. Anal. 102 (2023), no. 7, 1889-1913.
[10] Z. Dong, R. Zhang, T. Zhang, Ergodicity for stochastic conservation laws with multiplicative noise, Comm. Math. Phys. 400 (2023), no. 3, 1739-1789.
[11] T. Drivas, A. Dunlap, C. Graham, J. La, L. Ryzhik, Invariant measures for stochastic conservation laws on the line, Nonlinearity 36 (2023), no. 9, 4553-4584.
[12] J. Feng and D. Nualart, Stochastic Scalar Conservation Laws, J. Funct. Anal., **255** (2008) 313-373.
[13] B. Gess and P. Souganidis, Scalar conservation laws with multiple rough fluxes, Commun. Math. Sci. **13** (2015) 1569-1597.
[14] I. Gyöngy, Existence and uniqueness results for semilinear stochastic partial differential equations, Stochastic Process. Appl. **73** (1998) 271-299.
[15] I. Gyöngy and C. Rovira, On L^p-solutions of semilinear stochastic partial differential equations, Stochastic Process. Appl. **90** (2000) 83-108.
[16] M. Hofmanova, Degenerate parabolic stochastic partial differential equations, Stochastic Process. Appl. **123** (2013) 4294-4336.
[17] J. V. Kim, On a stochastic scalar conservation law, Indiana Univ. Math. J **52** (2003) 227-256.
[18] K. Kobayasi and D. Noboriguchi, A stochastic conservation law with nonhomogeneous Dirichlet boundary conditions, Acta Mathematica Vietnamica, 41 (2016), no. 4, 607-632.
[19] K. Kobayasi and D. Noboriguchi, Well-posedness for stochastic scalar conservation laws with the initial-boundary condition, J. Math. Anal. Appl. 461 (2018), no. 2, 1416-1458.
[20] N. Konatar, Uniqueness for stochastic scalar conservation laws on Riemannian manifolds revisited, Filomat 36 (2022), no. 5, 1615-1634.

[21] S. N. Kruzhkov, Generalized solutions of the Cauchy problem in the large for first order nonlinear equations, (Russian) Dokl. Akad. Nauk. SSSR **187** (1969) 29-32.
[22] S. N. Kruzhkov, First order quasilinear equation with several independent variables, Mat. Sb. (N.S.) **81** (1970) 228-255.
[23] A. Lunardi, Analytic Semigroup and Optimal Regularity in Parabolic Problems, Birk$\ddot{a}$user, Basel 1995.
[24] G. Lv, J. Duan, H. Gao and J. Wu, On a stochastic nonlocal conservation law in a bounded domain, Bull. Sci. Math., **140** (2016) 718-746.
[25] G. Lv, H. Gao, J. Wei, Kinetic solutions for nonlocal stochastic conservation laws, Fract. Calc. Appl. Anal. 24 (2021), no. 2, 559-584.
[26] G. Lv and J. Wu, Heterogeneous stochastic scalar conservation laws with non-homogeneous Dirichlet boundary conditions, J. Hyperbolic Differ. Equ. 15 (2018), no. 2, 291-328.
[27] C. Olivera, Well-posedness of the non-local conservation law by stochastic perturbation, Manuscripta Math. 162 (2020), no. 3-4, 367-387.
[28] F. Otto, Initial boundary-value problem for a scalar conservation law, C. R. Acad. Sci. Paris **322** (1996) 729-734.
[29] A. Porretta and J. Vovelle, L^1-solutions to first-order hyperbolic equations in bounded domains, Comm. Partial Differential Equations **28** (2003) 381-408.
[30] G. Vallet and P. Wittbold, On a stochastic first-order hyperbolic equation in a bounded domain, Infin. Dimens. Anal. Quantum Probab., **12** (2009) 1-39.
[31] Z. Wu, R. Zhang, Central limit theorem and moderate deviation principle for stochastic scalar conservation laws, J. Math. Anal. Appl. 516 (2022), no. 1, Paper No. 126445, 26 pp.
[32] E. Yu. Panov, On measure-valued solutions of the Cauchy problem for a first-order quasilinear equation, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996) 107-148.
[33] X. Zhang, Regularities for semilinear stochastic partial differential equations, J. Funct. Anal. **249** (2007) 454-476.