Filomat 39:9 (2025), 3099–3125
https://doi.org/10.2298/FIL2509099P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and non-existence of positive solutions for a class of fractional boundary value problems

S. Panigrahi[a,*], **Raghvendra Kumar**[a]

[a] *School of Mathematics & Statistics, University of Hyderabad, Hyderabad - 500 046 India*

Abstract. In this paper, we consider a boundary value problem consisting of the nonlinear fractional differential equation

$$-D^{\alpha}_{0^+}u + aD^{\gamma}_{0^+}u = f(t,u), \quad 0 < t < 1,$$

with nonlocal boundary conditions

$$D^{\beta}_{0^+}u(0) = 0, \quad D^{\alpha-\gamma}_{0^+}u(1) = au(1), \quad u'(1) = 0,$$

where, $2 < \gamma < \alpha \leq 3$, $0 \leq \beta < \alpha - \gamma$, $0 \leq a < \Gamma(\alpha - \gamma + 1)$ and $f(t,u) \in C([0,1] \times [0,\infty), [0,\infty))$ and $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative of order α. The associated Green's function is derived in terms of the generalized Mittag-Leffler functions and it is shown that it satisfies certain properties. An attempt has been made to establish the existence and non-existence of positive solutions by using Leggett-Williams fixed point theorems on a cone in a Banach space. The results obtained in this paper extended and generalizes the result of [R. Graef, L. Kong, Q. Kong and M. Wang, Positive solutions of nonlocal fractional boundary value problems, Discrete Contin. Dyn. Syst. 7 (4) (2013), 283–290]. Finally, we provide a couple of examples to illustrate the validation of established results.

1. Introduction

Fractional differential equations have a wide range of applications in various fields of science and engineering and play an important role in describing physics more accurately than classical integer order differential equations. Existence of positive solutions for nonlinear fractional boundary value problems with different boundary conditions have been studied by many researchers [2, 4, 7, 10] and the references therein. There are many techniques for solving fractional boundary value problems. If Riemann-Liouville fractional derivative is involve, the only viable way is to transform the boundary value problem into an

2020 *Mathematics Subject Classification*. Primary 34B15; Secondary 34B18.

Keywords. Continuous fractional calculus, Boundary value problem, Green's function, Positive solutions, Leggett-Williams fixed point theorem in cone, Banach space.

Received: 01 June 2023; Accepted: 19 January 2025

Communicated by Adrian Petrusel

The second author was supported by University Grant Commission Fellowship, India through ref. no. 201610152817 dated Feb 04, 2021.

* Corresponding author: S. Panigrahi

Email addresses: panigrahi2008@gmail.com (S. Panigrahi), raghvendrakumar0802@gmail.com (Raghvendra Kumar)

ORCID iDs: https://orcid.org/0000-0003-4704-5102 (S. Panigrahi), https://orcid.org/0009-0001-1721-4270 (Raghvendra Kumar)

integral equation, and use various fixed point theorems to find the fixed points of the respective operators. This idea is used by many researchers ; see, for example, [2, 4, 5, 7, 8, 10], and the references therein. The operators are constructed based on establishment of the associated Green's function. Furthermore, to apply fixed point theory, one usually require the positivity of the Green's function.

Finding the Green's function is a difficult task in case of fractional boundary value problem, but there are several papers dealing with existence of positive solutions for fractional boundary value problems with the help of properties of the Green's function and appropriate conditions on the nonlinear part of the differential equation. Bai and Lü [2] investigated the existence and multiplicity of positive solutions by using the Green's function for the nonlinear fractional boundary value problem:

$$-D^{\alpha}_{0^+}u + f(t,u) = 0, \quad 0 < t < 1, \tag{1}$$

and

$$u(0) = 0 = u(1), \tag{2}$$

where, $1 < \alpha \leq 2$ is a real number, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative of $h : [0,1] \to \mathbb{R}$ defined by

$$D^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}h(s)ds,$$

where $n = \lfloor\alpha\rfloor + 1$ and $\lfloor\alpha\rfloor$ is the integer part of α, provided that the integral on the right hand side exists.

By using Krasnosel'skii and Leggett-Williams fixed point theorems, along with growth conditions on f, they established the existence of at least one or three positive solutions respectively. By using [2, Lemma 2.2], Bai and Lü [2] constructed the Green's function for the boundary value problem

$$-D^{\alpha}_{0^+}u = 0, \quad 0 < t < 1, \tag{3}$$

and

$$u(0) = 0 = u(1), \tag{4}$$

by first taking the α-th integral of the equation

$$-D^{\alpha}_{0^+}u = h(t) \tag{5}$$

and determining the constants by using boundary conditions (4). This approach has also been employed by M. Feng et al. [4] and C. Goodrich [5] to construct the Green's functions for the problems consisting of equation (3) along with one of the boundary conditions

$$u^{(i)}(0) = D^{\nu}_{0^+}u(1) = 0, \quad 0 \leq i \leq N-2, 1 \leq \nu \leq N-2,$$

or

$$u^{(i)}(0) = 0, u(1) = \int_0^1 h(s)u(s)ds, \quad 0 \leq i \leq N-2, \alpha \leq N < \alpha+1,$$

where N is defined as in [2, Lemma 2.2].

However, all boundary conditions investigated so far involved the boundary condition $u(0) = 0$, which is vital in the construction of the corresponding Green's function. In fact, by [2, Lemma 2.2], a term with a negative power $C_N t^{\alpha-N}$ will appear after integrating equation (3). This term causes a singularity at $t = 0$ if

$C_N \neq 0$. Hence, $u(0) = 0$ is used to ensure that $C_N = 0$.

Using the idea used by Bai and Lü [2] and together with spectral theory, Graef et al. [7] investigated the boundary value problem consisting of the nonlinear fractional differential equation

$$-D^{\alpha}_{0^+}u + au = w(t)f(t,u), \quad 0 < t < 1, \tag{6}$$

and the integral boundary conditions

$$u(0) = 0, u(1) = I^{\alpha}_{0^+}u(1), \tag{7}$$

where, $1 < \alpha \leq 2, 0 < a < \gamma(\alpha + 1)$, $w \in L[0,1]$, $w(t) \geq 0$ a.e. on $[0,1]$ and $f \in C([0,1] \times [0,\infty), [0,\infty))$. Here, $I^{\alpha}_{0^+}$ is the α-th Riemann-Liouville integral of $h : [0,1] \to \mathbb{R}$ defined by

$$D^{-\alpha}_{0^+}h(t) = I^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

and Γ is the Gamma function.

In 2013, Graef et al. [6] studied the boundary value problem

$$-D^{\alpha}_{0^+}u + aD^{\gamma}_{0^+}u = f(t,u), \quad 0 < t < 1, \tag{8}$$

with the boundary conditions

$$D^{\beta}_{0^+}u(0) = 0, \quad D^{\alpha-\gamma}_{0^+}u(1) = au(1), \tag{9}$$

where, $1 < \gamma < \alpha \leq 2$, $0 \leq \beta < \alpha - \gamma$, $0 \leq a < \Gamma(\alpha - \gamma + 1)$ and $f(t,u) \in C([0,1] \times [0,\infty), [0,\infty))$.

Motivated by the above problems, we consider the boundary value problem consisting of the fractional differential equation

$$-D^{\alpha}_{0^+}u + aD^{\gamma}_{0^+}u = f(t,u), \quad 0 < t < 1, \tag{10}$$

with boundary conditions

$$D^{\beta}_{0^+}u(0) = 0, \quad D^{\alpha-\gamma}_{0^+}u(1) = au(1), \quad u'(1) = 0, \tag{11}$$

where, $2 < \gamma < \alpha \leq 3$, $0 \leq \beta < \alpha - \gamma$, $0 \leq a < \Gamma(\alpha - \gamma + 1)$ and $f(t,u) \in C([0,1] \times [0,\infty), [0,\infty))$.

In this paper, we first derive the corresponding Green's function in terms of the generalized Mittag-Leffler function. It is independent of β as long as β satisfies $0 \leq \beta < \alpha - \gamma, 0 \leq a < \Gamma(\alpha - \gamma + 1)$. Consequently, the problem (10) and (11) is converted to an equivalent Fredholm integral equation of the second kind. Finally, an attempt has been made to establish the existence of at least one or more positive solutions by using Leggett- William fixed point theorems on a cone in a Banach space for the fractional derivative of order α, where $2 < \gamma < \alpha \leq 3$. We would like to mentioned here that results obtained in this paper are new even in the case $\alpha = 3$, and extended and generalizes the results of [6].

The plan of the paper is as follows. In Section 2, we provide some elementary results concerning the fractional calculus and Leggett-Williams fixed point theorems. Section 3 deals with the construction of the Green's function and its various properties. In Section 4, we establish the existence of at least one or the multiplicity of positive solutions. Finally, in Section 5, we give a few examples to justify the results in the previous section.

2. Definitions and Preliminaries

For basics on fractional calculus, one can refer to the books [11], [14], [15]. The following basic results on fractional calculus are used to study the existence or multiplicity of positive solutions.

Definition 2.1. (*see* [13, Definition 2]) *Let X be a real Banach space. A nonempty closed, convex set $\mathscr{P} \subset X$ is called a cone if it satisfies the following two conditions:*
(i) If $u \in \mathscr{P}, \lambda \geq 0$ implies $\lambda u \in \mathscr{P}$;
(ii) If $u \in \mathscr{P}$, $-u \in \mathscr{P}$ implies $u = 0$, where 0 denotes the zero element of X.

Definition 2.2. (*see* [11, Equation 1.8.17]) *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\gamma}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \gamma)},$$

where $z, \gamma \in \mathbb{C}$ and $\Re(\alpha) > 0$. It is an entire function and hence it is convergent for all $z \in \mathbb{C}$.

Definition 2.3. (*see* [2, Definition 2.3]) *The map θ is said to be a nonnegative continuous concave functional on a cone $\mathscr{P}$ of a real Banach space X provided that $\theta : \mathscr{P} \to [0, \infty)$ is continuous and*

$$\theta(\lambda u + (1 - \lambda)v) \geq \lambda\theta(u) + (1 - \lambda)\theta(v),$$

for all $u, v \in \mathscr{P}$ and $0 \leq \lambda \leq 1$.

Remark 2.1. *As a basic example, for $\lambda > -1$ and $\alpha > 0$,*

$$D_{0^+}^{-\alpha} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\alpha + \lambda + 1)} t^{\alpha+\lambda}.$$

Similarly, for $\alpha > 0, \lambda > -1$,

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda-\alpha}.$$

Lemma 2.1. (*see* [2, Lemma 2.1]) *Let $\alpha > 0$ and $u \in C(0, 1) \cap L(0, 1)$. Then the solution of the differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

is given by $u(t) = \sum_{i=1}^{N} C_i t^{\alpha-i}$ for some $C_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, N$.

Lemma 2.2. (*see* [2, Lemma 2.2]) *Assume that $\alpha > 0$ and $u \in C(0, 1) \cap L(0, 1)$ has an α-th fractional derivative that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$D_{0^+}^{-\alpha}(D_{0^+}^{\alpha} u(t)) = u(t) + \sum_{i=1}^{N} C_i t^{\alpha-i};$$

for some $C_i \in \mathbb{R}, i = 1, 2, 3, \ldots, N$ and $\alpha \leq N < \alpha + 1$.

Note that Lemma 2.1 and Lemma 2.2 are crucial in finding an integral representation of the boundary value problem (10) and (11).

Lemma 2.3. (*see* [11, Lemma 2.3]) *If $\Re(\alpha) > 0$, $\Re(\gamma) > 0$, and $u \in L^p(0, t)$ $(1 \leq p \leq \infty)$, then*

$$D_{0^+}^{-\alpha}(D_{0^+}^{-\gamma} u(t)) = D_{0^+}^{-\gamma}(D_{0^+}^{-\alpha} u(t)) = D_{0^+}^{-(\alpha+\gamma)} u(t)$$

holds almost at every point in $[0, t]$.

Lemma 2.4. (*see* [11, Lemma 2.4]) *If* $\Re(\alpha) > 0$ *and* $u \in L^p(0,t)$ $(1 \le p \le \infty)$ *then the following relation*

$$D^{\alpha}_{0^+}(D^{-\alpha}_{0^+}u(t)) = u(t)$$

holds almost everywhere on $[0,t]$.

Lemma 2.5. (*see* [11, Property 2.2]) *If* $\Re(\alpha) > \Re(\beta) > 0$ *and* $u \in L^p(0,t)$ $(1 \le p \le \infty)$, *the relation*

$$D^{\beta}_{0^+}(D^{-\alpha}_{0^+}u(t)) = D^{-(\alpha-\beta)}_{0^+}u(t)$$

holds almost everywhere on $[0,t]$.

Lemma 2.6. (*see* [12, Theorem 7.3-1, page no. 375]) *Let* X *be Banach space and* $\mathscr{D} : X \to X$ *be a linear operator with the operator sup norm* $\|\mathscr{D}\|$ *and spectral radius* ρ *of* $\mathscr{D}$. *Then*
(a) $\rho(\mathscr{D}) \le \|\mathscr{D}\|$
(b) If $\rho(\mathscr{D}) < 1$, *then* $(I - \mathscr{D})^{-1}$ *exists and* $(I-\mathscr{D})^{-1} = \sum_{k=0}^{\infty} \mathscr{D}^k$, *where* I *stands for identity operator.*

In order to show the existence or multiplicity of positive solutions of the boundary value problem (10), (11), we will use the following Leggett-Williams fixed-point theorems [13].

Theorem 2.7. (*see* [13, Theorem 3.2]) *Let* $\mathscr{P}$ *be a cone in a real Banach space* X, $\mathscr{P}_c = \{u \in \mathscr{P} : \|u\| \le c\}$. *Suppose* $T : \overline{\mathscr{P}_c} \to \overline{\mathscr{P}}$ *is completely continuous and suppose there exists a concave positive functional* θ *on* $\mathscr{P}$ *with* $\theta(u) \le \|u\|$ *and for all* $u \in \mathscr{P}$ *and* $\mathscr{P}(\theta, a, b) = \{u \in \mathscr{P} : \theta(u) \ge a, \|u\| \le b\}$, *and there exist numbers* $0 < a < b \le c$ *satisfying the following conditions:*
(C1) $\{u \in \mathscr{P}(\theta,a,b) : \theta(u) > a\} \ne \phi$ *and* $\theta(Tu) > a$ *if* $u \in \mathscr{P}(\theta,a,b)$;
(C2) $Tu \in \mathscr{P}_c$ *if* $u \in \mathscr{P}(\theta,a,c)$;
(C3) $\theta(Tu) > a$, *for all* $u \in \mathscr{P}(\theta,a,c)$ *with* $\|Tu\| > b$.
Then T *has a fixed point* $u \in \mathscr{P}(\theta,a,b)$.

Theorem 2.8. (*see* [13, Theorem 3.5]) *Let* $\mathscr{P}$ *be a cone in a real Banach space* X, $\mathscr{P}_c = \{u \in \mathscr{P} : \|u\| \le c\}$. *Suppose* $T : \overline{\mathscr{P}_c} \to \overline{\mathscr{P}}$ *is completely continuous and suppose there exists a concave positive functional* θ *on* $\mathscr{P}$ *with* $\theta(u) \le \|u\|$, *and for all* $u \in \mathscr{P}$ *and* $\mathscr{P}(\theta,a,b) = \{u \in \mathscr{P} : \theta(u) \ge a, \|u\| \le b\}$, *and there exist numbers* $0 < a < b < c$ *satisfying the following conditions:*
(C4) $\{u \in \mathscr{P}(\theta,b,c) : \theta(u) > b\} \ne \phi$ *and* $\theta(Tu) > b$ *if* $u \in \mathscr{P}(\theta,b,c)$;
(C5) $\|Tu\| < a$ *if* $u \in \mathscr{P}_a$;
(C6) $\theta(Tu) > \frac{b}{c}\|Tu\|$ *for each* $u \in \mathscr{P}_c$ *such that* $\|Tu\| > c$.
Then T *has at least two fixed points in* $\mathscr{P}_c$.

Theorem 2.9. (*see* [13, Theorem 3.3]) *Let* $\mathscr{P}$ *be a cone in a real Banach space* X, $\mathscr{P}_c = \{u \in \mathscr{P} : \|u\| \le c\}$. *Suppose* $T : \overline{\mathscr{P}_c} \to \overline{\mathscr{P}_c}$ *is completely continuous and suppose there exists a concave positive functional* θ *on* $\mathscr{P}$ *with* $\theta(u) \le \|u\|$ *and* $\mathscr{P}(\theta,a,b) = \{u \in \mathscr{P} : \theta(u) \ge a, \|u\| \le b\}$, *and there exist numbers* $0 < d < a < b \le c$ *satisfying the following conditions:*
(C7) $\{u \in \mathscr{P}(\theta,a,b) : \theta(u) > a\} \ne \phi$ *and* $\theta(Tu) > a$ *if* $u \in \mathscr{P}(\theta,a,b)$;
(C8) $\|Tu\| < d$ *if* $u \in \mathscr{P}_d$;
(C9) $\theta(Tu) > a$, *for all* $u \in \mathscr{P}(\theta,a,c)$ *with* $\|Tu\| > b$.
Then T *has at least three fixed points* u_1, u_2, u_3 *with* $\|u_1\| < d$, $\theta(u_2) > a$, $\|u_3\| > d$, *and* $\theta(u_3) < a$.

Remark 2.2. (*see* [2, Remark 2.3])) *If* $b = c$, *then condition* (C_7) *implies condition* (C_9) *in Theorem 2.9.*

3. Green's Function and its Properties

In this section, we derive an integral representation of the solution in terms of the generalized Mittag-Leffler function for the linearized problem with fractional boundary conditions.

Lemma 3.1. *Let $E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\neq 0$. If $u\in C(0,1)\cap L(0,1)$ is a solution of*

$$-D^{\alpha}_{0^+}u+aD^{\gamma}_{0^+}u=h(t),\quad 0<t<1, \tag{12}$$

with boundary conditions

$$D^{\beta}_{0^+}u(0)=0,\quad D^{\alpha-\gamma}_{0^+}u(1)=au(1),\quad u'(1)=0, \tag{13}$$

where, $2<\gamma<\alpha\leq 3, 0\leq\beta<\alpha-\gamma$, $0\leq a<\Gamma(\alpha-\gamma+1)$, and $h\in C([0,1]\times[0,\infty))$. Then

$$u(t)=\int_0^1 G(t,s)h(s)ds\ ,$$

where,

$$G(t,s)=\begin{cases} G_1(t,s), & s\geq t,\\ G_1(t,s)-(t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}], & s\leq t,\end{cases} \tag{14}$$

where,

$$\begin{aligned} G_1(t,s)&=t^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\\ &\quad+t^{\alpha-1}E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]\left[(1-s)^{\gamma-1}-(\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right]\end{aligned}$$

and $G(t,s)$ is called the Green's function associated to the homogeneous boundary value problem (10), (11).

Proof. For any $h\in C[0,1]$, let u be a solution of boundary value problem (12), (13). On applying $D^{-\alpha}_{0^+}$ on both the sides of the equation (12), we obtain

$$-D^{-\alpha}_{0^+}(D^{\alpha}_{0^+}u)+aD^{-\alpha}_{0^+}(D^{\gamma}_{0^+}u)=D^{-\alpha}_{0^+}h(t),$$

that is,

$$-D^{-\alpha}_{0^+}(D^{\alpha}_{0^+}u)+aD^{-(\alpha-\gamma)}_{0^+}(D^{-\gamma}_{0^+}(D^{\gamma}_{0^+}u))=D^{-\alpha}_{0^+}h(t).$$

By using the Lemma 2.2, we obtain

$$\begin{aligned}&-[u(t)+\tilde{c}_1t^{\alpha-1}+\tilde{c}_2t^{\alpha-2}+\tilde{c}_3t^{\alpha-3}]+aD^{-(\alpha-\gamma)}_{0^+}(u(t))+aD^{-(\alpha-\gamma)}_{0^+}(\tilde{c}_4t^{\gamma-1})+aD^{-(\alpha-\gamma)}_{0^+}(\tilde{c}_5t^{\gamma-2})+aD^{-(\alpha-\gamma)}_{0^+}(\tilde{c}_6t^{\gamma-3})\\ &=D^{-\alpha}_{0^+}h(t).\end{aligned}$$

By Remark 2.1, this becomes

$$\begin{aligned}&u(t)+\tilde{c}_1t^{\alpha-1}+\tilde{c}_2t^{\alpha-2}+\tilde{c}_3t^{\alpha-3}-aD^{-(\alpha-\gamma)}_{0^+}(u(t))-a\tilde{c}_4\frac{\Gamma(\gamma)}{\Gamma(\alpha)}t^{\alpha-1}-a\tilde{c}_5\frac{\Gamma(\gamma-1)}{\Gamma(\alpha-1)}t^{\alpha-2}-a\tilde{c}_6\frac{\Gamma(\gamma-2)}{\Gamma(\alpha-2)}t^{\alpha-3}\\ &=-D^{-\alpha}_{0^+}h(t).\end{aligned}$$

Combining like power terms of t, we obtain

$$(I-aD^{-(\alpha-\gamma)}_{0^+})u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}-D^{-\alpha}_{0^+}h(t). \tag{15}$$

Taking $D^{\beta}_{0^+}$ on both sides of the equation (15) gives

$$D^{\beta}_{0^+}u(t) - aD^{\beta}_{0^+}(D^{-(\alpha-\gamma)}_{0^+}u(t)) = c_1D^{\beta}_{0^+}t^{\alpha-1} + c_2D^{\beta}_{0^+}t^{\alpha-2} + c_3D^{\beta}_{0^+}t^{\alpha-3} - D^{\beta}_{0^+}(D^{-\alpha}_{0^+}h(t)).$$

By Remark 2.1 Lemma 2.5, we obtain

$$D^{\beta}_{0^+}u(t) - aD^{-(\alpha-\beta-\gamma)}_{0^+}u(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}t^{\alpha-\beta-2} + c_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha-\beta-2)}t^{\alpha-\beta-3} - D^{-(\alpha-\beta)}_{0^+}u(t) \tag{16}$$

as $\alpha-\beta-\gamma > 0$, so $D^{-(\alpha-\beta-\gamma)}_{0^+}u(t)$ is a fractional integral, and $\alpha-\beta-3 \leq 0$; $\alpha - 3 \leq 0$ *implies* $\alpha-\beta-3 \leq -\beta \leq 0$; and $\alpha-\beta > 0$.

Since, $D^{\beta}_{0^+}u(0) = 0$ and the value of the fractional integrals $D^{-(\alpha-\beta-\gamma)}_{0^+}u(t), D^{-(\alpha-\beta)}_{0^+}u(t)$ at $t = 0$ is zero, from equation (16), we obtain

$$D^{\beta}_{0^+}u(t)\bigg|_{t=0} - aD^{-(\alpha-\beta-\gamma)}_{0^+}u(t)\bigg|_{t=0} = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}\bigg|_{t=0} + c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}t^{\alpha-\beta-2}\bigg|_{t=0} + c_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha-\beta-2)}t^{\alpha-\beta-3}\bigg|_{t=0} - D^{-(\alpha-\beta)}_{0^+}u(t)\bigg|_{t=0}$$

that is,

$$0 = c_1 \times 0 + c_2 \times 0 + c_3 \lim_{t\to 0} t^{\alpha-\beta-3} = c_3 \lim_{t\to 0} t^{\alpha-\beta-3} \;\; \textit{which implies} \;\; c_3 = 0.$$

Applying $D^{\alpha-\gamma}_{0^+}$ to both the sides of equation (15) and using Remark 2.1 and Lemmas 2.4 and 2.5, we obtain

$$D^{\alpha-\gamma}_{0^+}u(t) - au(t) = c_1D^{\alpha-\gamma}_{0^+}t^{\alpha-1} + c_2D^{\alpha-\gamma}_{0^+}t^{\alpha-2} - D^{\alpha-\gamma}_{0^+}(D^{-\alpha}_{0^+}h(t)).$$

That is,

$$D^{\alpha-\gamma}_{0^+}u(t) - au(t) = \frac{c_1\Gamma(\alpha)}{\Gamma(\gamma)}t^{\gamma-1} + \frac{c_2\Gamma(\alpha-1)}{\Gamma(\gamma-1)}t^{\gamma-2} - D^{-\gamma}_{0^+}h(t). \tag{17}$$

Using the boundary condition $D^{\alpha-\gamma}_{0^+}u(1) = au(1)$ in (17), we obtain

$$c_1(\alpha-1) + c_2(\gamma-1) = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\gamma-1}h(s)ds. \tag{18}$$

Let us define the map's $\mathscr{A}$ and $\mathscr{B}$ on Banach space X by

$$(\mathscr{A}u)(t) = a(D^{-(\alpha-\gamma)}_{0^+}u)(t) \;\; \textit{and} \quad (\mathscr{B}h)(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} - D^{-\alpha}_{0^+}h(t).$$

From equation (15), we have

$$(I - \mathscr{A})u(t) = (\mathscr{B}h)(t). \tag{19}$$

Since $0 \leq a < \Gamma(\alpha-\gamma+1)$,

$$\begin{aligned}
\|\mathscr{A}\| &= \sup_{\|u\|=1}\|\mathscr{A}u\| \\
&= \sup_{\|u\|=1}\|aD^{-(\alpha-\gamma)}_{0^+}u\| \\
&\leq \sup_{\|u\|=1}\left|\frac{a}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}u(s)ds\right| \\
&\leq \frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}ds \\
&= \frac{a}{\Gamma(\alpha-\gamma+1)} < 1.
\end{aligned}$$

From equation (19) and Lemma 2.6, we obtain

$$u(t) = \sum_{n=0}^{\infty} \mathscr{A}^n(\mathscr{B}h)(t),$$

that is,

$$u(t) = \sum_{n=0}^{\infty} \mathscr{A}^n[c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - D_{0^+}^{-\alpha} h(t)]. \tag{20}$$

Calculation of $\mathscr{A}^n(\mathscr{B}h)(t)$ gives

$$\mathscr{A}^n[c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - D_{0^+}^{-\alpha} h(t)] = c_1 \mathscr{A}^n t^{\alpha-1} + c_2 \mathscr{A}^n t^{\alpha-2} - \mathscr{A}^n(D_{0^+}^{-\alpha} h)(t).$$

The value of $\mathscr{A}^n(t^{\alpha-1})$, by using the Remark 2.1, is found to be

$$\begin{aligned}
\mathscr{A}^n(t^{\alpha-1}) &= \underbrace{\mathscr{A}\ldots\mathscr{A}}_{n times}(t^{\alpha-1}) \\
&= \underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(At^{\alpha-1}) \\
&= \underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(aD_{0^+}^{-(\alpha-\gamma)} t^{\alpha-1}) \\
&= a\underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(D_{0^+}^{-(\alpha-\gamma)} t^{\alpha-1}) \\
&= a\underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha-\gamma)} t^{2\alpha-\gamma-1}\right) \\
&= \frac{a^n \Gamma(\alpha)}{\Gamma((\alpha-\gamma)n+\alpha)} t^{(\alpha-\gamma)n+\alpha-1}.
\end{aligned} \tag{21}$$

Similarly, we have

$$\mathscr{A}^n(t^{\alpha-2}) = \frac{a^n \Gamma(\alpha-1)}{\Gamma((\alpha-\gamma)n+\alpha-1)} t^{(\alpha-\gamma)n+\alpha-2}. \tag{22}$$

The value of $\mathscr{A}^n(D_{0^+}^{-\alpha} h(t))$ by using Lemma 2.3 is

$$\begin{aligned}
\mathscr{A}^n(D_{0^+}^{-\alpha} h(t)) &= \underbrace{\mathscr{A}\ldots\mathscr{A}}_{n times}(D_{0^+}^{-\alpha} h(t)) \\
&= \underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(\mathscr{A} D_{0^+}^{-\alpha} h(t)) \\
&= a\underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(D_{0^+}^{-(\alpha-\gamma)} D_{0^+}^{-\alpha} h(t)) \\
&= a\underbrace{\mathscr{A}\ldots\mathscr{A}}_{(n-1) times}(D_{0^+}^{-(2\alpha-\gamma)} h(t)) \\
&= \frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha)} \int_0^t (t-s)^{(\alpha-\gamma)n+\alpha-1} h(s) ds.
\end{aligned} \tag{23}$$

Using the equations (21), (22) and (23), we obtain the expression for $\mathscr{A}^n(\mathscr{B}h)(t)$ to be

$$\begin{aligned}\mathscr{A}^n(\mathscr{B}h)(t) = &-\frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha)}\int_0^t (t-s)^{(\alpha-\gamma)n+\alpha-1}h(s)ds \\ &+ c_1\frac{a^n\Gamma(\alpha)}{\Gamma((\alpha-\gamma)n+\alpha)}t^{(\alpha-\gamma)n+\alpha-1} + c_2\frac{a^n\Gamma(\alpha-1)}{\Gamma((\alpha-\gamma)n+\alpha-1)}t^{(\alpha-\gamma)n+\alpha-2}.\end{aligned} \tag{24}$$

Using (24) in (20), we obtain

$$\begin{aligned}u(t) = &\, c_1\Gamma(\alpha)t^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n t^{(\alpha-\gamma)n}}{\Gamma((\alpha-\gamma)n+\alpha)} + c_2\Gamma(\alpha-1)t^{\alpha-2}\sum_{n=0}^{\infty}\frac{a^n t^{(\alpha-\gamma)n}}{\Gamma((\alpha-\gamma)n+\alpha-1)} \\ &- \int_0^t (t-s)^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{(\alpha-\gamma)n}}{\Gamma(\alpha-\gamma)n+\alpha}h(s)ds.\end{aligned} \tag{25}$$

By using the Mittag-Leffler function in (25), the expression for $u(t)$ is given by

$$u(t) = c_1t^{\alpha-1}\Gamma(\alpha)E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}] + c_2t^{\alpha-2}\Gamma(\alpha-1)E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] - \int_0^t (t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}]h(s)ds. \tag{26}$$

From (25), we obtain

$$\begin{aligned}u'(t) = &\, c_1\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-1)t^{n(\alpha-\gamma)+\alpha-2}}{\Gamma((\alpha-\gamma)n+\alpha)} + c_2\Gamma(\alpha-1)\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-2)t^{n(\alpha-\gamma)+\alpha-3}}{\Gamma((\alpha-\gamma)n+\alpha-1)} \\ &- \int_0^t\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-1)(t-s)^{n(\alpha-\gamma)+\alpha-2}}{\Gamma((\alpha-\gamma)n+\alpha)}h(s)ds.\end{aligned}$$

On applying the property of the Gamma function, namely $\Gamma(z) = (z-1)\Gamma(z-1)$, we obtain

$$\begin{aligned}u'(t) = &\, c_1\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-1)t^{n(\alpha-\gamma)+\alpha-2}}{((\alpha-\gamma)n+\alpha-1)\Gamma((\alpha-\gamma)n+\alpha-1)} + c_2\Gamma(\alpha-1)\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-2)t^{n(\alpha-\gamma)+\alpha-3}}{\Gamma((\alpha-\gamma)n+\alpha-2)\Gamma((\alpha-\gamma)n+\alpha-2)} \\ &- \int_0^t\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)+\alpha-1)(t-s)^{n(\alpha-\gamma)+\alpha-2}}{\Gamma((\alpha-\gamma)n+\alpha-1)\Gamma((\alpha-\gamma)n+\alpha-1)}h(s)ds.\end{aligned}$$

That is,

$$\begin{aligned}u'(t) = &\, c_1\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)+\alpha-2}}{\Gamma((\alpha-\gamma)n+\alpha-1)} + c_2\Gamma(\alpha-1)\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)+\alpha-3}}{\Gamma((\alpha-\gamma)n+\alpha-2)} \\ &- \int_0^t\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\gamma)+\alpha-2}}{\Gamma((\alpha-\gamma)n+\alpha-1)}h(s)ds.\end{aligned}$$

Expressing $u'(t)$ in terms of the Mittag-Leffler function gives

$$u'(t) = c_1t^{\alpha-2}\Gamma(\alpha)E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] + c_2t^{\alpha-3}\Gamma(\alpha-1)E_{\alpha-\gamma,\alpha-2}[at^{\alpha-\gamma}] - \int_0^t (t-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(t-s)^{\alpha-\gamma}]h(s)ds. \tag{27}$$

Using the boundary condition $u'(1) = 0$ in (27), we obtain

$$c_1\Gamma(\alpha)E_{\alpha-\gamma,\alpha-1}[a] + c_2\Gamma(\alpha-1)E_{\alpha-\gamma,\alpha-2}[a] = \int_0^1 (1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]h(s)ds. \tag{28}$$

On solving the equations (18) and (28), the values of c_1, c_2 are given by

$$c_1 = \frac{1}{\Gamma(\alpha)} I_1$$

where,

$$I_1 = \int_0^1 \left[(1-s)^{\gamma-1} - (\gamma-1) \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \right] h(s) ds$$

and

$$c_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 \left[\frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right] h(s) ds.$$

Therefore, the unique solution of the problem (12), (13) is

$$u(t) = \int_0^1 G(t,s) h(s) ds.$$

This completes the proof of the lemma. □

The following lemma gives the existence of the Green's function for the boundary value problem (10), (11).

Lemma 3.2. *The function $G(t,s)$ defined by* (14)*, is the Green's function for the boundary value problem* (10), (11).

Lemma 3.3. *Let $0 \le a < \Gamma(\alpha-\gamma+1)$, for $0 \le t \le 1$. Then the following hold:*

(a) $E_{\alpha-\gamma,\alpha}[a] < E_{\alpha-\gamma,\alpha-1}[a]$, for all $0 \le a < \Gamma(\alpha-\gamma+1)$.
(b) $E_{\alpha-\gamma,\alpha}[a]$ is monotonic increasing for $a \in [0, \Gamma(\alpha-\gamma+1))$.
(c) $E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]$ is monotonic increasing for $t \in [0,1]$.
(d) $E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}]$ is monotonic increasing for all t which satisfy the relation $0 \le s \le t \le 1$.
(e) $E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] \ge E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]$ for all $t \in [0,1]$.
(f) $(t-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}] \le (1-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}]$ for all t which satisfy the relation $0 \le s \le t \le 1$.

Proof. (a) We have

$$\begin{aligned}
E_{\alpha-\gamma,\alpha}[a] - E_{\alpha-\gamma,\alpha-1}[a] &= \sum_{n=0}^{\infty} \frac{a^n}{(\Gamma(\alpha-\gamma)n+\alpha)} - \sum_{n=0}^{\infty} \frac{a^n}{(\Gamma((\alpha-\gamma)n+\alpha-1))} \\
&= \sum_{n=0}^{\infty} a^n \left\{ \frac{1}{\Gamma((\alpha-\gamma)n+\alpha)} - \frac{1}{\Gamma((\alpha-\gamma)n+\alpha-1)} \right\} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha-1)} \left\{ \frac{1}{((\alpha-\gamma)n+\alpha-1)} - 1 \right\} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha-1)} \left\{ \frac{1-(\alpha-\gamma)n-\alpha+1}{((\alpha-\gamma)n+\alpha-1)} \right\} \\
&= \sum_{n=0}^{\infty} \frac{a^n(2-\alpha-(\alpha-\gamma)n)}{\Gamma((\alpha-\gamma)n+\alpha)}.
\end{aligned}$$

Since, $2-\alpha < 0, -n(\alpha-\gamma) < 0$, then $(2-\alpha) - n(\alpha-\gamma) < 0$. Thus,

$$E_{\alpha-\gamma,\alpha}[a] - E_{\alpha-\gamma,\alpha-1}[a] = \sum_{n=0}^{\infty} \frac{a^n(2-\alpha-(\alpha-\gamma)n)}{\Gamma((\alpha-\gamma)n+\alpha)} < 0.$$

Hence,

$$E_{\alpha-\gamma,\alpha}[a] < E_{\alpha-\gamma,\alpha-1}[a], \quad for \;\; all \;\; 0 \le a < \Gamma(\alpha-\gamma+1).$$

(b) Note that

$$E_{\alpha-\gamma,\alpha}[a] = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha)}. \tag{29}$$

Differentiating the equation (29) with respect to a, we obtain

$$\frac{d}{da}E_{\alpha-\gamma,\alpha}[a] = \frac{d}{da}\sum_{n=0}^{\infty} \frac{a^n}{\Gamma((\alpha-\gamma)n+\alpha)} = \sum_{n=1}^{\infty} \frac{na^{n-1}}{\Gamma((\alpha-\gamma)n+\alpha)} \ge 0.$$

Hence, $E_{\alpha-\gamma,\alpha}[a]$ is monotonic increasing for a on $[0, \Gamma(\alpha-\gamma+1))$.

(c) Note that,

$$E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] = \sum_{n=0}^{\infty} \frac{a^n t^{n(\alpha-\gamma)}}{\Gamma((\alpha-\gamma)n+\alpha-1)}.$$

Differentiating w.r.t. t, we obtain

$$\frac{\partial}{\partial t}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} \frac{a^n t^{n(\alpha-\gamma)}}{\Gamma((\alpha-\gamma)n+\alpha-1)} = \sum_{n=1}^{\infty} \frac{n(\alpha-\gamma)a^n t^{n(\alpha-\gamma)-1}}{\Gamma((\alpha-\gamma)n+\alpha-1)}.$$

Since $t^{n(\alpha-\gamma)-1} \ge 0$, for all $t \in [0,1]$ and $na^n(\alpha-\gamma) \ge 0$, then

$$\frac{\partial}{\partial t}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] \ge 0, \quad for \;\; all \quad t \in [0,1].$$

Hence, $E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]$ is monotonic increasing for $t \in [0,1]$.

(d) We have,

$$E_{\alpha-\gamma,\alpha-1}[a(t-s)^{\alpha-\gamma}] = \sum_{n=0}^{\infty} \frac{a^n (t-s)^{n(\alpha-\gamma)}}{\Gamma(n(\alpha-\gamma)+\alpha)}.$$

Differentiating w.r.t. t, we have

$$\frac{\partial}{\partial t}E_{\alpha-\gamma,\alpha-1}[a(t-s)^{\alpha-\gamma}] = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} \frac{a^n (t-s)^{n(\alpha-\gamma)}}{\Gamma(n(\alpha-\gamma)+\alpha)} = \sum_{n=1}^{\infty} \frac{n(\alpha-\gamma)a^n (t-s)^{n(\alpha-\gamma)-1}}{\Gamma(n(\alpha-\gamma)+\alpha)} \ge 0,$$

since each term in the summation is non-negative. Hence, $E_{\alpha-\gamma,\alpha-1}[a(t-s)^{\alpha-\gamma}]$ is monotonic increasing for all t such that $0 \le s \le t \le 1$.

(*e*) Note that,

$$\begin{aligned}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]-E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}] &= \sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)}}{\Gamma((n(\alpha-\gamma)+\alpha-1)}-\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)}}{\Gamma(n(\alpha-\gamma+\alpha))}\\ &= \sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)}}{\Gamma((n(\alpha-\gamma)+\alpha-1)}\left\{1-\frac{1}{n(\alpha-\gamma+\alpha)-1)}\right\}\\ &= \sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)}}{\Gamma((n(\alpha-\gamma)+\alpha-1)}\left\{\frac{n(\alpha-\gamma)+\alpha-2}{n(\alpha-\gamma+\alpha)-1)}\right\}\\ &= \sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\gamma)}(n(\alpha-\gamma)+\alpha-2)}{\Gamma((n(\alpha-\gamma)+\alpha)}\geq 0,\end{aligned}$$

since, for all $t\in[0,1]$, each term in the summation is non-negative. Hence,

$$E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\geq E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}],\quad for\ \ all\quad t\in[0,1].$$

(*f*) Note that,

$$(t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}]=\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\gamma)+\alpha-1}}{\Gamma(n(\alpha-\gamma)+\alpha)}.$$

Differentiating with respect to t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}] &= \frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\gamma)+\alpha-1}}{\Gamma(n(\alpha-\gamma)+\alpha)}=\sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)-1)(t-s)^{n(\alpha-\gamma)+\alpha-2}}{\Gamma(n(\alpha-\gamma)+\alpha)}\\ &= \sum_{n=0}^{\infty}\frac{a^n(n(\alpha-\gamma)-1)(t-s)^{n(\alpha-\gamma)+\alpha-2}}{(n(\alpha-\gamma)+\alpha-1)\Gamma(n(\alpha-\gamma)+\alpha-1)}\\ &= \sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\gamma)+\alpha-2}}{\Gamma(n(\alpha-\gamma)+\alpha-1)}\geq 0,\end{aligned}$$

since each term in summation is non-negative. Hence, for all t such that $0\leq s\leq t\leq 1$,

$$(t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}]\leq(1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}].$$

This completes the proof of the lemma. □

Next, we state the properties of the Green's function which will be subsequently used to prove our main results.

Lemma 3.4. *The Green's function $G(t,s)$ as defined in* (14) *with*

$$G_2(s)=\frac{G_4(s)}{G_5(s)}>0,$$

for all $s\in(0,1)$, where

$$\begin{aligned}G_4(s)&=(1-s(\gamma-1))\left\{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]\right\}\\ &\quad-\frac{\Gamma\alpha}{s^{\alpha-2}}\left\{(1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}](E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a])\right\}\end{aligned}$$

and

$$G_5(s)=E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a],$$

satisfies the following properties:

(i) $G(t,s) \le \overline{G}(s)$ for all $(t,s) \in [0,1]\times[0,1]$,
where,
$$\overline{G}(s) = E_{\alpha-\gamma,\alpha-1}[a]\left[1+\gamma\left\{\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] + (1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right\}\right].$$

(ii) $G(t,s) > G_3(s)$ for all $(t,s) \in (0,1)\times(0,1)$, where $G_3(s) = \dfrac{s^{\alpha-2}}{\Gamma(\alpha)} \times G_2(s)$.

(iii) Since $\overline{G}(s), G_3(s)$ are positive bounds, we conclude that $0 < G_3(s) < G(t,s) \le \overline{G}(s)$; hence $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$.

(iv) $G(t,1) = 0$ for all $t \in [0,1]$.

(v) $G(0,s) = 0$ for all $s \in [0,1]$.

(vi) $G(t,s)$ is a continuous function for all $t,s \in [0,1]$.

Proof. (i) To find the upper bound of $G(t,s)$, we will use $t^{\alpha-1} < 1, t^{\alpha-2} < 1$, and $(1-s)^{\gamma-1} < 1$, and properties (a), (c) and (e) of Lemma 3.3. From (14), we obtain $G(t,s) \le G_1(t,s)$ for all $t,s \in [0,1]$, where $G_1(t,s)$ is defined as in the Lemma 3.1. For $s \ge t$, from the Lemma 3.1, we have

$$G(t,s) = G_1(t,s) \le |G_1(t,s)|,$$

where

$$\begin{aligned}
G(t,s) &= G_1(t,s)\\
&\le \Bigg| t^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\\
&\quad + t^{\alpha-1}E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]\times\\
&\quad \left[(1-s)^{\gamma-1} - (\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right]\Bigg|\\
&\le \left|t^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\right|\left|\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right|\\
&\quad + \left|t^{\alpha-1}E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]\right|\times\\
&\quad \left|\left[(1-s)^{\gamma-1} - (\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right]\right|\\
&\le E_{\alpha-\gamma,\alpha-1}[a]\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid} + \frac{(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right]\\
&\quad + E_{\alpha-\gamma,\alpha-1}[a]\Bigg[(1-s)^{\gamma-1} + (\gamma-1)\left\{\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right\}\\
&\quad + (\gamma-1)\left\{\frac{(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right\}\Bigg]\\
&\le E_{\alpha-\gamma,\alpha-1}[a]\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid} + \frac{(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right]\\
&\quad + E_{\alpha-\gamma,\alpha-1}[a]\Bigg[(1-s)^{\gamma-1} + (\gamma-1)\left\{\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right\}\\
&\quad + (\gamma-1)\left\{\frac{E_{\alpha-\gamma,\alpha-1}[a]}{\mid E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\mid}\right\}\Bigg].
\end{aligned}$$

Combining all the similar terms for $(1-s)^{\alpha-2}$ and $(1-s)^{\gamma-1}$, we obtain

$$G(t,s) \leq E_{\alpha-\gamma,\alpha-1}[a]\left[1+\gamma\left\{\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]+(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{\left\{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]\right.}\right\}\right\}\right].$$

(ii) For $s \leq t$, we have

$$\begin{aligned}
G(t,s) &= G_1(t,s) - (t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}] \\
&= t^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] + t^{\alpha-1}E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]\times \\
&\quad \left[(1-s)^{\gamma-1}-(\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right] \\
&\quad - (t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}].
\end{aligned}$$

For $t \geq s$, we have $t^{\alpha-1} \geq s^{\alpha-1}$ and $t^{\alpha-2} \geq s^{\alpha-2}$ and using (e) and (f) of the Lemma 3.3, we obtain

$$\begin{aligned}
G(t,s) &\geq s^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}]\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] \\
&\quad + s^{\alpha-1}E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}]\times \\
&\quad \left[(1-s)^{\gamma-1}-(\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right] \\
&\quad - (t-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}] \\
&\geq \frac{s^{\alpha-2}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] + \frac{s^{\alpha-1}}{\Gamma(\alpha)}\times \\
&\quad \left[(1-s)^{\gamma-1}-(\gamma-1)\left(\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)\right] \\
&\quad - (1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}] \\
&= \frac{s^{\alpha-2}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] + \frac{s^{\alpha-1}(1-s)^{\gamma-1}}{\Gamma(\alpha)} \\
&\quad - \frac{(\gamma-1)s^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] \\
&\quad - (1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}] \\
&> \frac{s^{\alpha-2}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] \\
&\quad - \frac{(\gamma-1)s^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] - (1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}] \\
&= \frac{s^{\alpha-2}(1-s(\gamma-1))}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-2}E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}]-(1-s)^{\gamma-1}E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right] \\
&\quad - \frac{s^{\alpha-2}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^{\alpha-2}}\left(\frac{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a]-(\gamma-1)E_{\alpha-\gamma,\alpha-1}[a]}\right)(1-s)^{\alpha-1}E_{\alpha-\gamma,\alpha}[a(1-s)^{\alpha-\gamma}] \\
&= \frac{s^{\alpha-2}}{\Gamma(\alpha)}G_2(s).
\end{aligned}$$

Since, $G_2(s) > 0$ for all $s \in (0,1)$,

$$G(t,s) > \frac{s^{\alpha-2}}{\Gamma(\alpha)} G_2(s) > 0, \quad for \ \ all \quad (t,s) \in (0,1) \times (0,1).$$

Proof of (iii) follows from the (i) and (ii). Hence,

$$0 < G_3(s) < G(t,s) \leq \overline{G}(s), \quad for \ \ all \quad (t,s) \in (0,1) \times (0,1).$$

Proof of the (iv), (v) and the (vi) are trivial, and hence are omitted.

This completes the proof of the lemma. □

Now we consider the nonlinear boundary value problem (10), (11). Define an operator

$$\begin{aligned}
(Tu)(t) &= t^{\alpha-1} E_{\alpha-\gamma,\alpha}[at^{\alpha-\gamma}] \times \\
&\quad \int_0^1 \left[(1-s)^{\gamma-1} - (\gamma-1) \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \right] f(s,u(s)) ds \\
&\quad + t^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[at^{\alpha-\gamma}] \int_0^1 \left[\frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right] f(s,u(s)) ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t-s)^{\alpha-\gamma}] f(s,u(s)) ds \\
&= \int_0^1 G(t,s) f(s,u(s)) ds,
\end{aligned}$$

where $G(t,s)$ is defined as in Lemma 3.1. It is easy to observe that $u \in C[0,1]$ is a solution of the boundary value problem (10), (11) if and only if u is a fixed point of T. Let $t_1, t_2 \in [0,1]$. Then,

$$|(Tu)(t_1) - (Tu)(t_2)| = \left| \int_0^1 [G(t_1,s) - G(t_2,s)] f(s,u(s)) ds \right|.$$

Since $G(t,s)$ is continuous on $[0,1] \times [0,1]$, then for $\epsilon_1 > 0$, there exists a $\delta > 0$ such that

$$|G(t_1,s) - G(t_2,s)| < \epsilon_1, \quad whenever \quad | t_1 - t_2 | < \delta \quad for \ \ all \ \ s \in [0,1].$$

Hence,

$$\left| (Tu)(t_1) - (Tu)(t_2) \right| \leq \int_0^1 \left| G(t_1,s) - G(t_2,s) \right| \left| f(s,u(s)) \right| ds < \epsilon_1 \int_0^1 | f(s,u(s)) | ds .$$

Since $f(s,u(s))$ is continuous on $[0,1] \times [0,\infty)$, then exists a constant $M > 0$ such that $|f(s,u(s))| \leq M$ for $s \in [0,1]$. Hence,

$$\left| (Tu)(t_1) - (Tu)(t_2) \right| < M\epsilon_1.$$

Thus, Tu is a continuous operator for all $u \in C[0,1]$. Further,

$$\begin{aligned}
|(Tu)(t)| &= \left| \int_0^1 G(t,s) f(s,u(s)) ds \right| \leq \int_0^1 | G(t,s) || f(s,u(s)) | ds \leq \max_{0 \leq t \leq 1, 0 \leq s \leq 1} | G(t,s) | \int_0^1 | f(s,u(s)) | ds \\
&\leq \max_{0 \leq t \leq 1, 0 \leq s \leq 1} | G(t,s) | M.
\end{aligned}$$

Hence, T is a bounded operator for all $u \in C[0,1]$ and for all $t \in [0,1]$.

If $S \subset C[0,1]$ is a nonempty and bounded subset, then $T(S)$ is equicontinuous. Indeed, Let $\epsilon_1 > 0$. Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. So, there exists a $\delta > 0$ such that for all $t_1, t_2 \in [0,1]$,

$$\mid t_1 - t_2 \mid < \delta \ \ implies \ \ \mid G(t_1,s) - G(t_2,s) \mid < \epsilon_1.$$

Let us take any $u \in S$ and $0 \le t_1 < t_2 \le 1$; we obtain

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| = \Bigg| & t_2^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_2^{\alpha-\gamma}] \int_0^1 \Bigg[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Bigg] f(s,u(s)) ds \times \\
& + t_2^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_2^{\alpha-\gamma}] \times \\
& \int_0^1 \left[\frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right] f(s,u(s)) ds \\
& - \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_2-s)^{\alpha-\gamma}] f(s,u(s)) ds \\
& - t_1^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_1^{\alpha-\gamma}] \int_0^1 \Bigg[(1-s)^{\gamma-1} \\
& - (\gamma-1) \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Bigg] f(s,u(s)) ds \\
& - t_1^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_1^{\alpha-\gamma}] \times \\
& \int_0^1 \left[\frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right] f(s,u(s)) ds \\
& + \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_1-s)^{\alpha-\gamma}] f(s,u(s)) ds \Bigg| \\
\le \Bigg| & \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_1-s)^{\alpha-\gamma}] f(s,u(s)) ds \\
& - \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_2-s)^{\alpha-\gamma}] f(s,u(s)) ds \Bigg| + \Bigg| t_2^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_2^{\alpha-\gamma}] \times \\
& \int_0^1 \Bigg[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Bigg] f(s,u(s)) ds \\
& - t_1^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_1^{\alpha-\gamma}] \int_0^1 \Bigg[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Bigg] f(s,u(s)) ds \Bigg| \\
& + \Bigg| t_2^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_2^{\alpha-\gamma}] \times \\
& \int_0^1 \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Bigg] f(s,u(s)) ds
\end{aligned}$$

$$
\begin{aligned}
& - t_1^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_1^{\alpha-\gamma}] \times \\
& \int_0^1 \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big| \\
\leq & \Big| \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_1-s)^{\alpha-\gamma}] f(s,u(s)) ds - \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_2-s)^{\alpha-\gamma}] f(s,u(s)) ds \Big| \\
& + \Big| t_2^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_2^{\alpha-\gamma}] \int_0^1 \Big[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \\
& - t_1^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_1^{\alpha-\gamma}] \int_0^1 \Big[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big| \\
& + \Big| t_2^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_2^{\alpha-\gamma}] \times \\
& \int_0^1 \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds - t_1^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_1^{\alpha-\gamma}] \times \\
& \int_0^1 \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big| \\
\leq & \Big| \int_0^{t_1} \Big\{ (t_1-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_1-s)^{\alpha-\gamma}] - (t_2-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_2-s)^{\alpha-\gamma}] \Big\} f(s,u(s)) ds \Big| \\
& + \Big| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha-\gamma,\alpha}[a(t_2-s)^{\alpha-\gamma}] f(s,u(s)) ds \Big| + M (t_2^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_2^{\alpha-\gamma}] - t_1^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_1^{\alpha-\gamma}]) \times \\
& \Big| \int_0^1 \Big[(1-s)^{\gamma-1} - (\gamma-1) \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big| \\
& + M (t_2^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_2^{\alpha-\gamma}] - t_1^{\alpha-1} E_{\alpha-\gamma,\alpha}[a t_1^{\alpha-\gamma}]) \Big| \int_0^1 \Big[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big| \\
& + M (t_2^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_2^{\alpha-\gamma}] - t_1^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a t_1^{\alpha-\gamma}]) \Big| \int_0^1 \Big[(1-s)^{\gamma-1} - (\gamma-1) \times \\
& \left\{ \frac{(1-s)^{\alpha-2} E_{\alpha-\gamma,\alpha-1}[a(1-s)^{\alpha-\gamma}] - (1-s)^{\gamma-1} E_{\alpha-\gamma,\alpha-1}[a]}{E_{\alpha-\gamma,\alpha-2}[a] - (\gamma-1) E_{\alpha-\gamma,\alpha-1}[a]} \right\} \Big] f(s,u(s)) ds \Big|.
\end{aligned}
$$

Clearly, the right-hand side of the above inequality tends to zero as $t_1 \to t_2$. So, for any $t \in [0,1]$, the family $\{(Tu)(t) : u \in S\}$ is equicontinuous. Hence, T is continuous, bounded and equicontinuous, so by Arzelà-Ascoli theorem [3, 9], T is a completely continuous operator in $C[0,1]$. This leads to the following lemma.

Lemma 3.5. *The operator $T : \mathscr{P}_c \to \mathscr{P}$ is completely continuous.*

4. Main results

In this section, we focus on the study of existence and nonexistence of positive solutions to the nonlinear boundary value problem (10), (11) by using the Leggett-Williams fixed point theorems.

Let $X = (C[0,1], \|.\|)$, where $\|u\| = \sup_{0\le t\le 1} |u(t)|$. Then X is a Banach space. Define the cone $\mathscr{P} \subset X$ by

$$\mathscr{P} = \{u \in C[0,1] : u(t) \ge 0, \quad for \;\; all \;\; t \in [0,1]\}.$$

Let the nonnegative continuous concave functional θ on a cone $\mathscr{P}$ be defined by

$$\theta(u) = \min_{0\le t\le 1} |u(t)|.$$

We have the completely continuous operator $T : \mathscr{P}_c \to \mathscr{P}$ defined by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) ds,$$

where $G(t,s)$ is defined as in Lemma 3.1. Property (iii) of Lemma 3.4 is also satisfied. Note that the fixed points of T are solutions of the boundary value problem (10), (11).

Theorem 4.1. *Suppose $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$ and there exist constants $0 < a < b \le c$, such that the following assumptions hold:*

(H_1) $\quad f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} a$, *for all* $(t,u) \in [0,1]\times[a,b]$;

(H_2) $\quad f(t,u(t)) \le \left(\int_0^1 \overline{G}(s)ds\right)^{-1} c$, *for all* $(t,u) \in [0,1]\times[a,c]$;

(H_3) $\quad f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} a$, *for all* $(t,u) \in [0,1]\times[a,c]$.

Then, the boundary value problem (10), (11) *has at least one positive solution u with $\theta(u) \ge a$ and $\|u\| \le c$.*

Proof. We shall show that all the hypothesis of Theorem 2.7 are satisfied. Let us take $u(t) = d \in (a,b)$ for some $t \in [0,1]$. Clearly, $u(t) \in \mathscr{P}(\theta,a,b)$ and $\theta(d) > d$; hence $\{u \in \mathscr{P}(\theta,a,b) : \theta(u) > a\} \neq \phi$. If $u \in \mathscr{P}(\theta,a,b)$ then $u(t) \in [a,b]$, so by (H_1),

$$\begin{aligned}\theta(Tu) &= \min_{0\le t\le 1}(Tu)(t)\\ &= \min_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\ &> \min_{0\le t\le 1}\int_0^1 G_3(s)\left(\int_0^1 G_3(s)ds\right)^{-1} a \;\; ds\\ &> a.\end{aligned}$$

Hence, $\theta(Tu) > a$.
Given $u \in \mathscr{P}(\theta,a,c)$ this implies $\theta(u) \ge a, \|u\| \le c$ which implies $u(t) \in [a,c]$ and by using (H_2), we obtain

$$\begin{aligned}\|Tu\| &= \sup_{\|u\|=1}\int_0^1 G(t,s)f(s,u(s))ds\\ &\le \int_0^1 \overline{G}(s)\left(\int_0^1 \overline{G}(s)ds\right)^{-1} c \;\; ds \le c.\end{aligned}$$

Hence, $Tu \in \mathscr{P}_c$.
Further, we have $u \in \mathscr{P}(\theta,a,c)$, so $u(t) \in [a,c]$, and by using (H_3), we obtain

$$\begin{aligned}\theta(Tu) &= \min_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\ &> \min_{0\le t\le 1}\int_0^1 G_3(s)\left(\int_0^1 G_3(s)ds\right)^{-1} a \;\; ds\\ &> a.\end{aligned}$$

Hence, all the hypothesis of Theorem 2.7 are satisfied. Therefore, the boundary value problem (10), (11) has at least one positive solution with $\theta(u) \geq a, \|u\| \leq c$. □

Theorem 4.2. *Suppose $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$ and there exist constants a,b,c with $0 < a < b < c$ such that the following assumptions holds:*

(H_4) $f(t,u(t)) < \left(\int_0^1 \overline{G}(s)ds\right)^{-1} a,$ *for all* $(t,u) \in [0,1]\times[0,a]$;

(H_5) $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} b,$ *for all* $(t,u) \in [0,1]\times[0,c]$;

(H_6) $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} b,$ *for all* $(t,u) \in [0,1]\times[b,c]$.

Then the boundary value problem (10), (11) *has at least two positive solutions* u_1, u_2 *with* $\|u_1\| \leq c, \|u_2\| \leq c$.

Proof. Let us define a positive concave functional

$$\theta(u) = \min_{0\leq t\leq 1} u(t).$$

We will show that all the hypothesis of Theorem 2.8 are satisfied. We have a completely continuous operator $T : \mathscr{P}_c \to \mathscr{P}$ defined by $(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds$, where $G(t,s)$ is defined as in Lemma 3.1 and satisfies the property (iii) of Lemma 3.4.

Take $u(t) = \frac{b+c}{2}$ for some $0 \leq t \leq 1$; clearly, this $u(t) \in \mathscr{P}(\theta,b,c)$. Then $\theta(u) = \theta\Big(\frac{b+c}{2}\Big) = \frac{b}{2} + \frac{c}{2} > b$. Hence, the set $\{u \in \mathscr{P}(\theta,b,c) : \theta(u) > b\} \neq \phi$.

Since $u \in \mathscr{P}(\theta,b,c), u(t) \in [b,c]$. Using assumption (H_6), we obtain

$$\begin{aligned}\theta(Tu) = \min_{0\leq t\leq 1}(Tu)(t) &= \min_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s))ds \\ &> \int_0^1 G_3(s)\Big(\int_0^1 G_3(s)ds\Big)^{-1} b \ \ ds \\ &= b.\end{aligned}$$

Hence, $\theta(Tu) > b$, for all $(t,u) \in [0,1]\times[b,c]$.

Since $(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds$, using assumption (H_4), we obtain

$$\begin{aligned}\|Tu\| &= \sup_{\|u\|=1}\int_0^1 G(t,s)f(s,u(s))ds \\ &< \int_0^1 \overline{G}(s)\Big(\int_0^1 \overline{G}(s)ds\Big)^{-1} a \quad ds \\ &= a.\end{aligned}$$

Hence, $\|Tu\| < a$, for all $t \in [0,1]$. Further, by assumption (H_5), we have

$$\begin{aligned}\theta(Tu) &= \min_{0\leq t\leq 1}(Tu)(t) = \min_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s))ds \\ &> \min_{0\leq t\leq 1}\int_0^1 G_3(s)\Big(\int_0^1 G_3(s)ds\Big)^{-1} b \ \ ds \\ &= b.\end{aligned}$$

Hence, $\theta(Tu) > b$ for all $(t,u) \in [0,1]\times[0,c]$. All the hypothesis of Theorem 2.8 are satisfied, so the boundary value problem (10), (11) has at least two positive solution u_1, u_2 with $\|u_1\| \le c,\ \|u_2\| \le c$. □

Theorem 4.3. *Suppose $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$ and there exist constants $0 < d < a < b < c$ such that the following assumptions hold:*

(H_8) $f(t,u(t)) < \left(\int_0^1 \overline{G}(s)ds\right)^{-1} d$, *for all* $(t,u) \in [0,1]\times[0,d]$;

(H_9) $f(t,u(t)) \le \left(\int_0^1 \overline{G}(s)ds\right)^{-1} b$, *for all* $(t,u) \in [0,1]\times[0,b]$;

(H_{10}) $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} a$, *for all* $(t,u) \in [0,1]\times[a,b]$.

Then the boundary value problem (10), (11) *has at least three positive solution u_1, u_2 and u_3 with $\|u_1\| < d, \theta(u_2) > a, \|u_3\| > d,\ \theta(u_3) < a$.*

Proof. Define a positive concave functional on a cone $\mathscr{P}$, with

$$\theta(u) = \min_{0\le t\le 1} u(t).$$

We have,

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds,$$

where $G(t,s)$ is defined as in the Lemma 3.1. On using the assumption (H_9), one can show that the operator $T : \mathscr{P}_b \to \mathscr{P}_b$ is well defined and

$$\begin{aligned}\|Tu\| &= \sup_{\|u\|=1}\int_0^1 G(t,s)f(s,u(s))ds\\ &\le \int_0^1 \overline{G}(s)\Big(\int_0^1 \overline{G}(s)ds\Big)^{-1} b\ \ ds\\ &\le b.\end{aligned}$$

Hence, $\|Tu\| \le b$ implies that $Tu \in \mathscr{P}_b$.

Let us take $u(t) = \dfrac{b+c}{2}$ for some $0 \le t \le 1$. Clearly, $u \in \mathscr{P}(\theta,a,b)$, and it is easy to verify that $\theta(u) > a$. Hence $\{u \in \mathscr{P}(\theta,a,b) : \theta(u) > a\} \ne \phi$. By (H_{10}), we obtain

$$\begin{aligned}\theta(Tu) = \min_{0\le t\le 1}(Tu)(t) &= \min_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\ &> \int_0^1 G_3(s)\Big(\int_0^1 G_3(s)ds\Big)^{-1} a\ \ ds\\ &= a.\end{aligned}$$

and by (H_8), we obtain

$$\begin{aligned}\|Tu\| &= \sup_{\|u\|=1}\int_0^1 G(t,s)f(s,u(s))ds\\ &< \int_0^1 \overline{G}(s)\Big(\int_0^1 \overline{G}(s)ds\Big)^{-1} d\ \ ds\\ &= d.\end{aligned}$$

By using the Remark 2.2 one can conclude that condition (C_7) implies condition (C_9) in Theorem 2.9. All the hypothesis of Theorem 2.9 are satisfied, hence the boundary value problem (10), (11) has at least three positive solutions u_1, u_2 and u_3 with $\|u_1\| < d, \theta(u_2) > a, \|u_3\| > d$ with $\theta(u_3) < a$. □

Theorem 4.4. *Suppose* $f(t,u) < u(t)\Big(\int_0^1 \overline{G}(s)ds\Big)^{-1}$ *for all* $(t,u) \in [0,1] \times [0,\infty)$, *then boundary value problem* (10), (11) *has no positive solution.*

Proof. Assume that the boundary value problem (10), (11) has a positive solution u with $\|u\| = r$, so there exists $t_1 \in [0,1]$ such that $u(t_1) = r$. Then

$$\begin{aligned}(Tu)(t_1) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &< u(t_1)\int_0^1 \overline{G}(s)\Big(\int_0^1 \overline{G}(s)ds\Big)^{-1} ds.\end{aligned}$$

If boundary value problem (10), (11) has a solution, then it must be fixed point of T, so

$$u(t_1) < u(t_1) \;\; \textit{implies} \;\; r < r,$$

which is a contradiction. Hence, the boundary value problem (10), (11) has no positive solution. □

5. Examples

Example 5.1. *Consider the fractional differential equation*

$$-D_{0^+}^{2.8}u + 0.1D_{0^+}^{2.1}u = f(t,u), \quad 2 < \gamma < \alpha \leq 3, \quad 0 < t < 1, \tag{30}$$

with boundary conditions

$$D_{0^+}^{\beta}u(0) = 0, \quad D_{0^+}^{0.7}u(1) = au(1), \quad u'(1) = 0, \tag{31}$$

where, $0 \leq \beta < 0.7$, *and* $f(t,u) = \Big(\frac{t}{100} + 0.2\Big)^2 e^{\frac{u^3 - 0.01}{2}} \sqrt{\cos u^{100}}$.
The positive lower bound of the Green's function $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (30), (31) *is given by*

$$G_3(s) = \frac{s^{0.8}}{1.6}G_2(s) = \frac{s^{0.8}}{1.6}\frac{G_4(s)}{G_5(s)},$$

where

$$\begin{aligned}G_4(s) &= (1-1.1s)\Big\{(1-s)^{0.8}E_{0.7,1.8}[0.1(1-s)^{0.7}] - (1-s)^{1.1}E_{0.7,1.8}[0.1]\Big\} \\ &\quad - \frac{1.6}{s^{0.8}}\Big\{(1-s)^{1.8}E_{0.7,2.8}[0.1(1-s)^{0.7}](E_{0.7,0.8}[0.1] - (1.1)E_{0.7,1.8}[0.1])\Big\},\end{aligned}$$

and

$$G_5(s) = E_{0.7,0.8}[0.1] - (1.1)E_{0.7,1.8}[0.1].$$

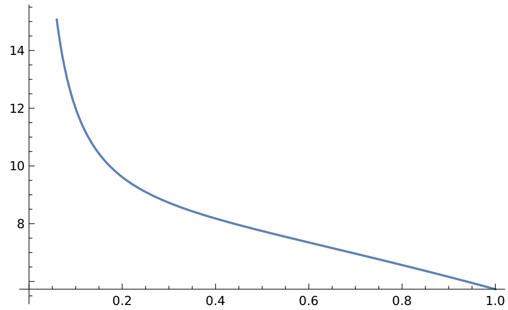

Figure 1: $G_2(s)$ is positive for $\alpha = 2.8, \gamma = 2.1$ and $a = 0.1$

The upper bound of the Green's function $G(t,s)$ corresponding to the homogeneous boundary value problem (30), (31) *is given by*

$$\overline{G}(s) = E_{0.7,1.8}[0.1]\left[1 + 2.1\left\{\frac{(1-s)^{0.8}E_{0.7,1.8}[0.1(1-s)^{0.7}] + (1-s)^{1.1}E_{0.7,1.8}[0.1]}{|\,E_{0.7,0.8}[0.1] - (1.1)E_{0.7,1.8}[0.1]\,|}\right\}\right].$$

We also have $\left(\int_0^1 G_3(s)ds\right)^{-1} \approx 0.416$ *and* $\left(\int_0^1 \overline{G}(s)ds\right)^{-1} \approx 0.136$ *and choosing positive constants* $a = 0.01$, $b = c = 0.2$, *we see that*

$$\begin{aligned} f(t,u) = \left(\frac{t}{100} + 0.2\right)^2 e^{\frac{u^3 - 0.01}{2}} \sqrt{\cos u^{100}} &\leq (0.01 + 0.2)^2 e^{\frac{(0.2)^3 - 0.01}{2}} \sqrt{\cos (0.2)^{100}} \\ &= 0.016 \\ &\leq (0.136 \times 0.2) \approx 0.0272. \end{aligned}$$

Hence, $f(t,u(t)) \leq \left(\int_0^1 \overline{G}(s)ds\right)^{-1} c \approx 0.0272$ *for all* $(t,u) \in [0,1] \times [0.01, 0.2]$ *and*

$$\begin{aligned} f(t,u) = \left(\frac{t}{100} + 0.2\right)^2 e^{\frac{u^3 - 0.01}{2}} \sqrt{\cos u^{100}} &\geq (0.2)^2 e^{\frac{(0.01)^3 - 0.01}{2}} \sqrt{\cos (0.01)^{100}} \\ &= 0.0397 \\ &> (0.136 \times 0.01) \\ &= 0.00136. \end{aligned}$$

Hence, $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} a \approx 0.00136$ *for all* $(t,u) \in [0,1] \times [0.01, 0.2]$. *All the hypothesis of Theorem 4.1 are satisfied, hence the boundary value problem* (30), (31) *has at least one positive solution with* $\theta(u) \geq 0.01$ *and* $\|u\| \leq 0.2$.

Example 5.2. *Consider the fractional boundary value problem*

$$-D_{0^+}^{2.909}u + 0.162 D_{0^+}^{2.002}u = f(t,u), \quad 2 < \gamma < \alpha \leq 3, \quad 0 < t < 1, \tag{32}$$

and

$$D_{0^+}^{\beta}u(0) = 0, \quad D_{0^+}^{0.907}u(1) = 0.162u(1), \quad u'(1) = 0, \tag{33}$$

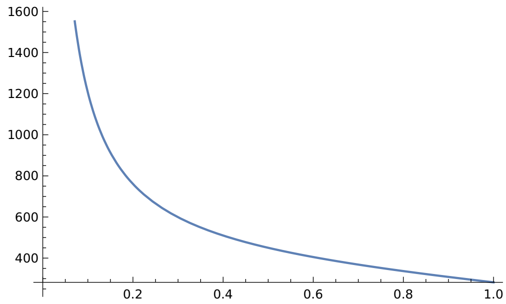

Figure 2: $G_2(s)$ is positive for $\alpha = 2.909, \gamma = 2.002$ and $a = 0.162$

where, $0 \leq \beta < 0.907$, *and*
$f(t,u) = \left(\frac{t}{100} + 0.1\right)^2 e^{\frac{u^2 - 0.412}{2}} \left(\cos^{14} u^{10}\right)\left(\ln(e+t)^{\frac{e^2}{2}}\right)$. *The positive lower bound of the Green's function* $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (32), (33) *is given by*

$$G_3(s) = \frac{s^{0.909}}{1.84} G_2(s) = \frac{s^{0.909}}{1.84} \frac{G_4(s)}{G_5(s)},$$

where

$$\begin{aligned} G_4(s) = (1 - 1.909s)&\Big\{(1-s)^{0.909} E_{0.907,1.909}[0.162(1-s)^{0.907}] - (1-s)^{1.002} E_{0.907,0.109}[0.162]\Big\} \\ &- \frac{1.84}{s^{0.909}}\Big\{(1-s)^{1.909} E_{0.907,2.909}[0.162(1-s)^{0.907}](E_{0.907,0.909}[0.162] - (1.002)E_{0.907,1.909}[0.162])\Big\}, \end{aligned}$$

and

$$G_5(s) = E_{0.907,0.909}[0.162] - (1.002)E_{0.907,1.909}[0.162].$$

The upper bound of the Green's function $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (32), (33) *is given by*

$$\begin{aligned} \overline{G}(s) = E_{0.907,1.909}[0.162] \times& \\ &\left[1 + 2.002\left\{\frac{(1-s)^{0.909} E_{0.907,1.909}[0.162(1-s)^{0.907}] + (1-s)^{1.002} E_{0.907,1.909}[0.162]}{\mid E_{0.907,0.909}[0.162] - (1.002)E_{0.907,1.909}[0.162] \mid}\right\}\right]. \end{aligned}$$

Also, we have $\left(\int_0^1 G_3(s)ds\right)^{-1} \approx 0.0081$ *and* $\left(\int_0^1 \overline{G}(s)ds\right)^{-1} \approx 0.0056$. *Let us choose the positive constants* $a = 0.1, b = 0.102$ *and* $c = 0.105$, *so that*

$$\begin{aligned} f(t,u) &= \left(\frac{t}{100} + 0.1\right)^2 e^{\frac{u^2 - 0.412}{2}} \left(\cos^{14} u^{10}\right)\left(\ln(e+t)^{\frac{e^2}{2}}\right) \\ &\leq \left(0.01 + 0.1\right)^2 e^{\frac{(0.1)^2 - 0.412}{2}} \left(\cos^{14}(0.1)^{10}\right)\left(\ln(e+1)^{\frac{e^2}{2}}\right) \\ &= 0.0000098 \\ &< (0.0056) \times (0.1) \\ &= 0.00056. \end{aligned}$$

Hence, $f(t,u(t)) < \left(\int_0^1 \overline{G}(s)ds\right)^{-1} a$*, for all* $(t,u) \in [0,1]\times[0,0.1]$*. Note that*

$$\begin{aligned} f(t,u) &= \left(\frac{t}{100}+0.1\right)^2 e^{\frac{u^2-0.412}{2}}\left(\cos^{14}u^{10}\right)\left(\ln(e+t)^{\frac{e^2}{2}}\right)\\ &\geq (0.1)^2 e^{\frac{-0.412}{2}}\left(\ln^2(e)^{\frac{e^2}{2}}\right)\\ &= 0.00813\\ &> (0.0081)\times(0.102) = 0.0008262. \end{aligned}$$

Hence, $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} b$*, for all* $(t,u) \in [0,1]\times[0,0.105]$ *and*

$$\begin{aligned} f(t,u) &= \left(\frac{t}{100}+0.1\right)^2 e^{\frac{u^2-0.412}{2}}\left(\cos^{14}u^{10}\right)\left(\ln(e+t)^{\frac{e^2}{2}}\right)\\ &\geq (0.1)^2 e^{\frac{(0.105)^2-0.412}{2}}\left(\cos^{14}(0.105)^{10}\right)\left(\ln^2(e)^{\frac{e^2}{2}}\right)\\ &= 0.0037\\ &> (0.0081)\times(0.102)\\ &= 0.0008262. \end{aligned}$$

So, $f(t,u(t)) > \left(\int_0^1 G_3(s)ds\right)^{-1} b$*, for all* $(t,u) \in [0,1]\times[0.102,0.105]$*. All the hypothesis of Theorem 4.2 are satisfied, hence the boundary value problem* (32), (33) *has at least two positive solutions with* $\|u_1\| \leq 0.105$ *and* $\|u_2\| \leq 0.105$*.*

Example 5.3. *Consider the fractional boundary value problem*

$$-D_{0^+}^{2.691}u + 0.1D_{0^+}^{2.001}u = f(t,u), \quad 2<\gamma<\alpha\leq 3, \quad 0<t<1, \tag{34}$$

and

$$D_{0^+}^{\beta}u(0) = 0, \quad D_{0^+}^{0.69}u(1) = 0.1u(1), \quad u'(1) = 0, \tag{35}$$

where $0 \leq \beta < 0.69$ *and*

$$f(t,u) = \begin{cases} \dfrac{t^2}{1000} + 4u^5 \quad , \quad 0\leq u\leq 1 \\[2ex] \dfrac{7}{2} + \dfrac{t^2}{1000} + \dfrac{(u)^{\frac{1}{1000}}}{2} \quad , \quad u\geq 1 \end{cases}$$

The positive lower bound of the Green's function $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (34), (35) *is given by,*

$$G_3(s) = \frac{s^{0.691}}{1.53}G_2(s) = \frac{s^{0.691}}{1.53}\frac{G_4(s)}{G_5(s)},$$

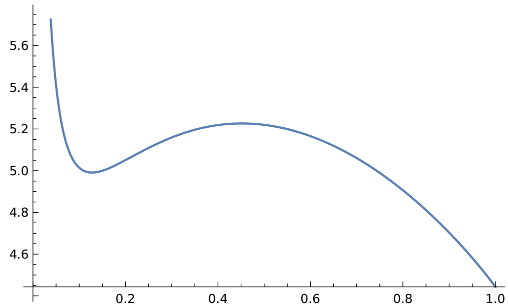

Figure 3: $G_2(s)$ is positive for $\alpha = 2.691, \gamma = 2.001$ and $a = 0.1$

where

$$G_4(s) = (1-1.001s)\Big\{(1-s)^{0.691}E_{0.69,1.691}[0.1(1-s)^{0.69}] - (1-s)^{1.001}E_{0.69,1.691}[0.1]\Big\}$$
$$-\frac{1.53}{s^{0.691}}\Big\{(1-s)^{1.691}E_{0.69,2.691}[0.1(1-s)^{0.69}](E_{0.69,0.691}[0.1] - (1.001)E_{0.69,1.691}[0.1])\Big\},$$

and

$$G_5(s) = E_{0.69,0.691}[0.1] - (1.001)E_{0.69,1.69}[0.1].$$

The upper bound of the Green's function $G(t,s)$ corresponding to the homogeneous boundary value problem of (34), (35) *is given by,*

$$\overline{G}(s) = E_{0.69,1.691}[0.1]\times$$
$$\left[1 + 2.001\left\{\frac{(1-s)^{0.691}E_{0.69,1.691}[0.1(1-s)^{0.69}] + (1-s)^{1.001}E_{0.69,1.691}[0.1]}{|\,E_{0.69,0.691}[0.1] - (1.001)E_{0.69,1.691}[0.1]\,|}\right\}\right].$$

Also, we have $\left(\int_0^1 G_3(s)ds\right)^{-1} \approx 0.536$ *and* $\left(\int_0^1 \overline{G}(s)ds\right)^{-1} \approx 0.134$, *and here take the positive constants* $d = 0.2, a = 1$ *and* $b = 25$, *so that*

$$\begin{aligned} f(t,u) = \frac{t^2}{1000} + 4u^5 &\leq \frac{1}{1000} + 4(0.5)^5 \\ &= 0.0028 \\ &\leq (0.134)(0.2) \\ &= 0.0268. \end{aligned}$$

This shows that $f(t,u) \leq \left(\int_0^1 \overline{G}(s)ds\right)^{-1} d$ *for all* $(t,u) \in [0,1]\times[0,0.2]$. *Note that*

$$\begin{aligned} f(t,u) = \frac{7}{2} + \frac{t^2}{1000} + \frac{(u)^{\frac{1}{1000}}}{2} &\geq \frac{7}{2} + \frac{1}{2} \\ &= 4 \\ &> (0.536)(1) \\ &= 0.536. \end{aligned}$$

Hence, $f(t,u) > \left(\int_0^1 G_3(s)ds\right)^{-1} a$, *for all* $(t,u) \in [0,1] \times [1,25]$.
Also we have,

$$f(t,u) = \frac{7}{2} + \frac{t^2}{1000} + \frac{(u)^{\frac{1}{1000}}}{2} \le f(t,u) = \frac{7}{2} + \frac{1}{1000} + \frac{(25)^{\frac{1}{1000}}}{2}$$
$$= 3.0026$$
$$\le 3.35.$$

So one can conclude that $f(t,u) \le \left(\int_0^1 \overline{G}(s)ds\right)^{-1} b$ *for all* $(t,u) \in [0,1] \times [0,25]$. *All the hypothesis of Theorem 4.3 are satisfied, hence the boundary value problem* (34), (35) *has at least three positive solutions with* $\|u_1\| < 0.2$, $\theta(u_2) > 1$, *and* $\|u_3\| > 0.2$ *with* $\theta(u_3) < 1$.

Example 5.4. *Consider the fractional boundary value problem*

$$-D_{0^+}^{2.78}u + 0.2D_{0^+}^{2.14}u = f(t,u), \quad 2 < \gamma < \alpha \le 3, \quad 0 < t < 1, \tag{36}$$

and

$$D_{0^+}^{\beta}u(0) = 0, \quad D_{0^+}^{0.64}u(1) = 0.2u(1), \quad u'(1) = 0, \tag{37}$$

where $0 \le \beta < 0.64$, *and* $f(t,u) = \frac{u^2 t}{10(u+1)}$. *The positive lower bound of the Green's function* $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (36), (37) *is given by*

$$G_3(s) = \frac{s^{0.78}}{1.64}G_2(s) = \frac{s^{0.78}}{1.64}\frac{G_4(s)}{G_5(s)},$$

where

$$G_4(s) = (1 - 1.14s)\left\{(1-s)^{0.78}E_{0.64,1.78}[0.2(1-s)^{0.64}] - (1-s)^{1.14}E_{0.64,1.78}[0.2]\right\}$$
$$- \frac{0.78}{s^{0.78}}\left\{(1-s)^{1.78}E_{0.64,2.78}[0.2(1-s)^{0.64}](E_{0.64,0.78}[0.2] - (1.14)E_{0.64,1.78}[0.2])\right\}$$

and

$$G_5(s) = E_{0.64,0.78}[0.2] - (1.14)E_{0.64,1.78}[0.2].$$

The upper bound of the Green's function $G(t,s)$ *corresponding to the homogeneous boundary value problem of* (36), (37) *is given by,*

$$\overline{G}(s) = E_{0.64,1.78}[0.2]\times$$
$$\left[1 + 2.14\left\{\frac{(1-s)^{0.78}E_{0.64,1.78}[0.2(1-s)^{0.64}] + (1-s)^{1.14}E_{0.64,1.78}[0.2]}{|\ E_{0.64,0.78}[0.2] - (1.78)E_{0.64,1.78}[0.2]\ |}\right\}\right].$$

Also, we have $\left(\int_0^1 G_3(s)ds\right)^{-1} \approx 0.425$ *and* $\left(\int_0^1 \overline{G}(s)ds\right)^{-1} \approx 0.125$, *so*

$$\frac{f(t,u)}{u} = \frac{ut}{10(u+1)} < 0.1 < 0.125.$$

Hence, $f(t,u) < u(t)\left(\int_0^1 \overline{G}(s)ds\right)^{-1}$ *for all* $(t,u) \in [0,1] \times [0,\infty))$. *All the hypothesis of Theorem 4.4 are satisfied, so the boundary value problem* (36), (37) *has no positive solution.*

6. Conclusion

In this paper, we extend the results in [6] to $2 < \gamma < \alpha \leq 3$, dealing with existence and nonexistence of positive solutions of fractional differential equation with fractional boundary conditions. The main tool used in this paper is Leggett-Williams fixed point theorem on a cone in a Banach space. In Lemma 3.4, we have shown that the Green's function is positive by use of Lemma 3.3. Theorem 4.1, Theorem 4.2 and Theorem 4.3 provide existence of at least one, at least two and at least three positive solutions respectively. Theorem 4.4 provides the existence of no positive solutions. Finally, examples illustrate the validity of the results.

References

[1] M. Ackerman, M. Wang and G. Yost, A fractional boundary value problem with separated boundary conditions, *Pan Amer. Math. J.* **25** (2) (2015), 1-12.

[2] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311** (2) (2005), 495–505.

[3] J. B. Conway, *A Course in Functional Analysis*, Germany: Springer, 1990.

[4] M. Feng, X. Zhang and W. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, *Bound. Value Probl.* **2011** (2011), 1-20.

[5] C. S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* **23** (9) (2010), 1050-1055.

[6] R. Graef, L. Kong, Q. Kong and M. Wang, Positive solutions of nonlocal fractional boundary value problems, *Discrete Contin. Dyn. Syst.* **7** (4) (2013), 283–290.

[7] R. Graef, L. Kong, Q. Kong and M. Wang, Fractional boundary value problems with integral boundary conditions, *Appl. Anal.* **92** (10) (2013), 2008-2020.

[8] R. Graef, L. Kong, Q. Kong and M. Wang, A fractional boundary value problem with a Dirichlet boundary condition, *Commun. Appl. Anal.* **19** (4) (2015), 497–504.

[9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[10] E. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2008** (3) (2008) , 1-11.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, Netherlands, 2006.

[12] E. Kreyszig, *Introductory Functional Analysis With Applications*, John Wiley and Sons, Inc., New York, 1978.

[13] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* **28** (4) (1979), 673–688.

[14] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, Wiley, New York, 1993.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, London, 1999.