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# Caputo-hybrid Hermite-Hadamard and Newton's type inequalities in multiplicative calculus with applications

**Muhammad Umar[a], Saad Ihsan Butt[a,*]**

[a]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan*

**Abstract.** The study addresses the need for a comprehensive mathematical framework for multiplicative (geometric) $P$-convex functions, specifically in the context of $P_{cap}$ (Proportional Caputo-Hybrid) operators. The research identifies a significant gap in existing literature concerning the formulation of specific inequalities and their applications to this class of functions. This study aims to fill the gap by developing and presenting new $H_rH_d$ (Hermite-Hadamard) type inequalities tailored to multiplicative $P$-convex functions using $P_{cap}$ operators. The lack of a detailed understanding and established results in this area underscores the importance of the study. The main contributions include the derivation of novel inequalities that extend the traditional concepts of convexity into a multiplicative framework. Additionally, a new Newton's type identity applicable to multiplicatively $P$-differentiable functions is introduced, which provides fresh insights and tools for analysis in this domain. The practical implications of the findings are demonstrated through applications to special means and type-1 modified Bessel functions. These applications not only validate the theoretical results, but also highlight their versatility and relevance in broader mathematical contexts. Research significantly advances the theoretical understanding of multiplicative $P$-convex functions, offering new analytical tools and frameworks. This advancement has potential implications for various mathematical and applied fields, including optimization and numerical analysis. The study suggests that future research could explore additional applications and extend the theoretical framework to other types of functions and operators, thereby broadening the scope and impact of the findings in this emerging area of study.

## 1. Introduction

Convexity theory has proven invaluable in many fields of mathematics and engineering, providing a thorough and cohesive framework for examining a wide range of issues. The exploration of convexity in conjunction with integral inequalities represents a captivating area of research and has a close relationship in the development of the theory of inequalities, which is an important tool in the study of some properties of solutions of differential equations as well as in the error estimates of quadrature formulas. In the subsequent, we provide the basic terminologies relevant to convexities and inequalities.

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* Corresponding author: Saad Ihsan Butt
*Email addresses:* umarqureshi987@gmail.com (Muhammad Umar), saadihsanbutt@gmail.com (Saad Ihsan Butt)
ORCID iDs: https://orcid.org/0000-0001-9911-1111 (Muhammad Umar), 0000-0001-7192-8269 (Saad Ihsan Butt)

**Definition 1.1.** *[26] The mapping $f : \mathrm{I} \subset \Re \rightarrow \Re$ is designated as convex, if*

$$f(ut + (1-t)v) \leq tf(u) + (1-t)f(v), \tag{1}$$

$\forall\ t \in [0,1]$ *and* $u, v \in \mathrm{I}$ *holds.*

**Definition 1.2.** *[26] A positive function $f : \mathrm{I} \rightarrow \Re$ is designated as logarithmically convex or multiplicatively convex, if*

$$f(ut + (1-t)v) \leq [f(u)]^t [f(v)]^{(1-t)}, \tag{2}$$

$\forall\ t \in [0,1]$ *and* $u, v \in \mathrm{I}$ *holds.*

The inequalities for convex functions identified by J. Hadamard and C. Hermite hold substantial importance inside the written works, see ( [11, 17] ( [30], p.137)). Such kind of inequalities states that if $f : \mathrm{I} \rightarrow \Re$ is possessing convexity on $\mathrm{I} \subseteq \Re$ and $u, v \in \mathrm{I}$ with $u < v$, then

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(\mathrm{x}) \mathrm{d}(\mathrm{x}) \leq \frac{f(u) + f(v)}{2}. \tag{3}$$

In the case of a concave function, the inequalities are satisfied in the reversed direction. Notably, Hadamard's inequality can be viewed as an improvement on the concept of convexity and Jensen's inequality makes it easy to derive. Over the past two decades, there has been a substantial focus on obtaining new bounds for both the right-hand and left-hand sides of inequality (3).

### 1.1. Multiplicative Calculus

Between 1967 and 1970, Grossman and Katz introduced a novel approach to derivatives and integrals, wherein they replaced subtraction and addition with division and multiplication. This innovation led to the development of a new calculus known as multiplicative calculus or non-Newtonian calculus. Despite effectively addressing various calculus-related issues, multiplicative calculus has not gained the same level of popularity as Newton and Leibniz's calculus. Its application is relatively limited, mainly encompassing positive functions. This raises the question of whether it is rational to create a specialized tool when a more comprehensive one already exists. The analogy can be drawn to mathematicians employing a polar coordinate system alongside the rectangular coordinate system, as each serves its purpose effectively([15, 36]).

Next we initiate by revisiting certain definitions, properties, and concepts related to differentiation, along with exploring aspects of multiplicative integration.

**Definition 1.3.** *[5] Assume that the function $f : \Re \rightarrow \Re$ is positive. The function $f$ denotes multiplicative derivative by $\mathrm{f}^*$ is defined as follows.*

$$\frac{\mathrm{d}^* f}{\mathrm{d}\varkappa} = \mathrm{f}^*(\varkappa) = \lim_{\mathrm{h} \rightarrow 0} \left(\frac{f(\varkappa + \mathrm{h})}{f(\varkappa)}\right)^{\frac{1}{\mathrm{h}}}.$$

**Remark 1.4.** *If $f$ has positive values and is differentiable at $\varkappa$, then $\mathrm{f}^*$ exists and the relation between $\mathrm{f}^*$ and ordinary derivative $\mathrm{f}'$ is as follows:*

$$\mathrm{f}^* = \mathrm{e}^{(\ln f(\varkappa))'} = \mathrm{e}^{\frac{\mathrm{f}'(\varkappa)}{f(\varkappa)}}.$$

The multiplicative derivative admits the following properties:

**Proposition 1.5.** *[5] Let $f$ and $\mathrm{g}$ be multiplicatively differentiable functions and $\mathrm{c}$ is arbitrary constant. Then functions $cf$, $f\mathrm{g}$, $\mathrm{f}/\mathrm{g}$ and $f + \mathrm{g}$ are $*$ differentiable*

1. $(\mathrm{cf})^*(\varkappa) = \mathrm{f}^*(\varkappa)$

2. $(\mathrm{fg})^*(\varkappa) = \mathrm{f}^*(\varkappa)\mathrm{g}^*(\varkappa)$
3. $(\mathrm{f}+\mathrm{g})^*(\varkappa) = \mathrm{f}^*(\varkappa)^{\frac{f(\varkappa)}{f(\varkappa)+\mathrm{g}(\varkappa)}}\mathrm{g}^*(\varkappa)^{\frac{\mathrm{g}(\varkappa)}{f(\varkappa)+\mathrm{g}(\varkappa)}}$
4. $\left(\frac{f}{\mathrm{g}}\right)^*(\varkappa) = \frac{\mathrm{f}^*(\varkappa)}{\mathrm{g}^*(\varkappa)}$
5. $(f^{\mathrm{g}})^*(\varkappa) = \mathrm{f}^*(\varkappa)^{\mathrm{g}(\varkappa)}f(\varkappa)^{\mathrm{g}'(\varkappa)}$

The multiplicative integral, occasionally referred to as the $^*$ integral, is symbolised by the symbol $\int_u^v (f(\varkappa))^{\mathrm{d}\varkappa}$. This mathematical model proposed by Bashirov et al. in [5]. In defining the classical Riemann integral of $f$ over the interval $[u, v]$ the approach involves employing the sum of product term, The definition of the multiplicative integral of $f$ over the interval $[u, v]$ involves raising the product of terms to a power.

The relationship that exists between the multiplicative integral and the Rimann integral is as outlined below [5]:

**Proposition 1.6.** *Riemann integrability of $f$ on $[\mathrm{u}, \mathrm{v}]$ implies the multiplicative integrability of $f$ on the same interval.*

$$\int_u^v (f(\varkappa))^{\mathrm{d}\varkappa} = \mathrm{e}^{\int_u^v \ln(f(\varkappa))\mathrm{d}\varkappa}.$$

Furthermore, as demonstrated by Bashirov et al. [5], multiplicative integrable has the subsequent characteristics and outcomes:

**Proposition 1.7.** *Riemann integrability of $f$ on $[\mathrm{u}, \mathrm{v}]$ implies * integrability of $f$ on the same interval*

1. $\int\limits_u^v ((f(\varkappa))^{\mathrm{p}})^{\mathrm{d}\varkappa} = \int\limits_u^v \left((f(\varkappa))^{\mathrm{d}\varkappa}\right)^{\mathrm{p}}$
2. $\int\limits_u^v (f(\varkappa)\,\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa} = \int\limits_u^v (f(\varkappa))^{\mathrm{d}\varkappa} . \int\limits_u^v (\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}$
3. $\int\limits_u^v \left(\frac{f(\varkappa)}{\mathrm{g}(\varkappa)}\right)^{\mathrm{d}\varkappa} = \frac{\int_u^v (f(\varkappa))^{\mathrm{d}\varkappa}}{\int_u^v (\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}}$
4. $\int\limits_u^v (f(\varkappa))^{\mathrm{d}\varkappa} = \int\limits_u^{\mathrm{c}} (f(\varkappa))^{\mathrm{d}\varkappa} . \int\limits_{\mathrm{c}}^v (f(\varkappa))^{\mathrm{d}\varkappa}, \mathrm{u} \le \mathrm{c} \le v$
5. $\int\limits_u^u (f(\varkappa))^{\mathrm{d}\varkappa} = 1, \int\limits_u^v (f(\varkappa))^{\mathrm{d}\varkappa} = \left(\int\limits_v^u (f(\varkappa_1))^{\mathrm{d}\varkappa}\right)^{-1}.$

Alternatively, the subsequent multiplicative Riemann-Liouville fractional integrals were put forward by Abdeljawed and Grossman [1].

**Definition 1.8.** *The symbol $({}_u\mathrm{I}_*^{\beta_o} f)(\varkappa)$ is a designation of multiplicative left Riemann-Liouville fractional integral of order $\beta_o \in \mathbb{C}, \mathrm{Re}(\beta_o) > 0$ with $\beta_o$ as an initial point is given by*

$$({}_u\mathrm{I}_*^{\beta_o} f)(\varkappa) = \mathrm{e}^{(\mathrm{J}_{u^+}^{\beta_o}(\ln\circ f))(\varkappa)},$$

*and what defines the multiplicative right one is*

$$({}_*\mathrm{I}_v^{\beta_o} f)(\varkappa) = \mathrm{e}^{(\mathrm{J}_{v^-}^{\beta_o}(\ln\circ f))(\varkappa)}.$$

*Here $\mathrm{J}_{v^-}^{\beta_o} f$ and $\mathrm{J}_{u^+}^{\beta_o} f$ describe the right and left Riemann-Liouville fractional integral, given by [19]*

$$\mathrm{J}_{v^-}^{\beta_o} f(t) = \frac{1}{\Gamma(\beta_o)}\int_{\varkappa}^{v} (t-\varkappa)^{\beta_o-1} f(t)\mathrm{d}t, \;\; v > \varkappa.$$

*and*

$$J^{\beta_o}_{u^+}f(t) = \frac{1}{\Gamma(\beta_o)}\int_u^{\varkappa}(\varkappa - t)^{\beta_o-1}f(t)dt,\ u < \varkappa$$

*accordingly, where* $\Gamma(\beta_o) = \int_0^\infty e^{-c}u^{\beta_o-1}du$. *Here is* $J^0_{v^-}f(\varkappa) = f(\varkappa) = J^0_{u^+}f(\varkappa)$.

**Theorem 1.9.** *(Multiplicative integration by parts [5]) Let* $f : [u,v] \to \Re$ *and* $g : [u,v] \to \Re$ *are possessing* $^*$ *differentiability and differentiability respectively, so the function* $f^g$ *is* $^*$ *integrable then it implies that*

$$\int_u^v \left(f^*(\varkappa)^{g(\varkappa)}\right)^{d\varkappa} = \frac{f(v)^{g(v)}}{f(u)^{g(u)}}.\frac{1}{\int_u^v \left(f(\varkappa)^{g'(\varkappa)}\right)^{d\varkappa}}.$$

**Lemma 1.10.** *[3] Let* $f : [u,v] \to \Re$ *and* $g : [u,v] \to \Re$ *are possessing* $^*$ *differentiability and differentiability respectively, so the function* $f^g$ *is* $^*$ *integrable then it implies that*

$$\int_u^v \left(f^*(h(\varkappa))^{h'(\varkappa)g(\varkappa)}\right)^{d\varkappa} = \frac{f(v)^{g(v)}}{f(u)^{g(u)}}.\frac{1}{\int_u^v \left(f(h(\varkappa))^{g'(\varkappa)}\right)^{d\varkappa}}.$$

Let us now provide some essential definitions and mathematical preliminaries of multiplicative calculus theory that will be used throughout this paper.

**Proposition 1.11.** *The* log *convexity of f and* g, *implies the* log *convexity of* fg *and* $\frac{f}{g}$.

For convex functions, the standard $H_rH_d$ inequality is provided by the equality (3).

In [2], Ali et al. demonstrated the $H_rH_d$ inequality for multiplicatively convex functions in the following way:

**Theorem 1.12.** *Let the positive function f is possessing multiplicatively convexity on* $[u,v]$ *then the subsequent disparities are true.*

$$f\left(\frac{u+v}{2}\right) \leq \left(\int_u^v (f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v-u}} \leq G(f(u), f(v)), \tag{4}$$

*where* $G(.,.)$ *is geometric mean.*

However, the $H_rH_d$ inequality for multiplicative Riemann-Liouville fractional integrals was demonstrated in [6], which is a noteworthy inequality.

**Theorem 1.13.** *Let the positive function f is possessing multiplicatively convexity on* $[u,v]$, *then, we obtain the subsequent* $H_rH_d$ *inequality for multiplicative fractional integrals of Riemann-Liouville*

$$f\left(\frac{u+v}{2}\right) \leq \left[{}_uI^{\beta_o}_* f(v)._*I^{\beta_o}_v f(u)\right]^{\frac{\Gamma(\beta_o+1)}{2(v-u)^{\beta_o}}} \leq G(f(u), f(v)), \tag{5}$$

*where* $G(.,.)$ *stands for geometric mean.*

**Theorem 1.14.** *[6] Let the positive function f is possessing multiplicatively convexity on* $[u,v]$, *then, we obtain the subsequent* $H_rH_d$ *inequality for multiplicative fractional integrals of Riemann-Liouville*

$$f\left(\frac{u+v}{2}\right) \leq \left[{}_{\frac{u+v}{2}}I^{\beta_o}_* f(v)._*I^{\beta_o}_{\frac{u+v}{2}} f(u)\right]^{\frac{2^{\beta_o-1}\Gamma(\beta_o+1)}{(v-u)^{\beta_o}}} \leq G(f(u), f(v)), \tag{6}$$

*where* $G(.,.)$ *stands geometric mean.*

The following definition is very important for multiplicative calculus.

**Definition 1.15.** *The multiplicative left latest proportional caputo hybrid operator of order $\beta_o \in \mathbb{C}$ designated by $({}_u\mathrm{D}_*^{\beta_o} f)(\varkappa)$ with $\mathrm{Re}(\beta_o) > 0$ by assuming $\beta_o$ as an initial point is given by*

$$({}_u\mathrm{D}_*^{\beta_o} f)(\varkappa) = \mathrm{e}^{({}^{\mathrm{PC}}_{\mathrm{b}^-}\mathrm{D}^{\beta_o}_{u^+}(\ln\circ f))(\varkappa)}$$

*and what defines the multiplicative right one is*

$$({}_*\mathrm{D}_v^{\beta_o} f)(\varkappa) = \mathrm{e}^{({}^{\mathrm{PC}}_{u^+}\mathrm{D}^{\beta_o}_{v^-}(\ln\circ f))(\varkappa)},$$

**Definition 1.16.** *[33] Let $\beta_o > 0$ and $\beta_o \notin \{1, 2, ...\}$, $\mathrm{n} = [\beta_o] + 1$, $f \in \mathrm{AC}^{\mathrm{n}}[u, v]$, the space of functions having* n−th *derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives possessing order $\beta_o$ are described as below:*

$$^{\mathrm{C}}\mathrm{D}^{\beta_o}_{v^-} f(t) = \frac{1}{\Gamma(\mathrm{n} - \beta_o)} \int_{\varkappa}^{v} (t - \varkappa)^{\mathrm{n}-\beta_o-1} f^{\mathrm{n}}(t)\mathrm{d}t,\ \varkappa < v.$$

*and*

$$^{\mathrm{C}}\mathrm{D}^{\beta_o}_{u^+} f(t) = \frac{1}{\Gamma(\mathrm{n} - \beta_o)} \int_{u}^{\varkappa} (\varkappa - t)^{\mathrm{n}-\beta_o-1} f^{\mathrm{n}}(t)\mathrm{d}t,\ \varkappa > u$$

*If $\beta_o = \mathrm{n} \in 1, 2, ...$ and usual derivative $f^{\mathrm{n}}(t)$ of order* n *exists, then Caputo fractional derivative $^{\mathrm{C}}\mathrm{D}^{\beta_o}_{u^+} f(t)$ overlaps with $f^{\mathrm{n}}(t)$ whereas $^{\mathrm{C}}\mathrm{D}^{\beta_o}_{v^-} f(t)$ with exactness to a constant multiplier $(-1)^{\mathrm{n}}$. In particular we achieve,*

$$^{\mathrm{C}}\mathrm{D}^{0}_{u^+} f(t) = {}^{\mathrm{C}}\mathrm{D}^{0}_{v^-} f(t) = f(t)$$

*where* $\mathrm{n} = 1$ *and* $\beta_o = 0$.

Within the topic of fractional calculus, one commonly utilized fractional derivative operator as the Caputo derivative operator. In terms of time, it may be defined as the fractional derivative of a function whose order is non integer.

In fractional calculus Caputo operator is widely used along with its derivative operator. It is define as the fractional derivative of a function with respect to time, where the order of the derivative is a non-integer value. The Riemann-Liouville integral operator, on the other hand, is a fractional integral operator that is also commonly used in fractional calculus. Several studies have been done in literature pertaining a variety of fractional integral operators concerning $H_rH_d$'s integral inequalities. The recent latest $P_{cap}$ is a mathematical operation that has been proposed as a non-local and singular operator, incorporating both derivative and integral operator components in its definition. It can be written as a simple linear combination of the operators for the Caputo derivative and the Riemann-Liouville integral [4, 16].

**Definition 1.17.** *[4, 16] Let the function $f : \mathrm{I} \subset \Re^+ \to \Re$ is possessing differentiability on $\mathrm{I}^\circ$ and $f, \mathrm{f}' \in \mathrm{L}^1(\mathrm{I})$. Then the $P_{cap}$ may be described as*

$$^{\mathrm{PC}}_{0}\mathrm{D}^{\beta_o}_{t} f(t) = \frac{1}{\Gamma(1 - \beta_o)} \int_{0}^{t} (\mathrm{K}_1(\beta_o, \varkappa) f(\varkappa) + \mathrm{K}_0(\beta_o, \varkappa)\mathrm{f}'(\varkappa))(t - \varkappa)^{-\beta_o}\mathrm{d}\varkappa,$$

*where $\beta_o \in [0, 1]$ and $\mathrm{K}_0$ and $\mathrm{K}_1$ are function satisfying*

$$\lim_{\beta_o \to 0^+} \mathrm{K}_0(\beta_o, \varkappa) = 0;\ \lim_{\beta_o \to 1} \mathrm{K}_0(\beta_o, \varkappa) = 1;\ \mathrm{K}_0(\beta_o, \varkappa) \neq 0,\ \beta_o \in (0, 1],$$

$$\lim_{\beta_o \to 0^+} \mathrm{K}_1(\beta_o, \varkappa) = 0;\ \lim_{\beta_o \to 1^-} \mathrm{K}_1(\beta_o, \varkappa) = 0;\ \mathrm{K}_1(\beta_o, \varkappa) \neq 0,\ \beta_o \in [0, 1).$$

Next we define the latest $P_{cap}$ of order $\beta_o$:

**Definition 1.18.** *[4, 16] Let the function $f : \mathrm{I} \subset \Re^+ \to \Re$ is possessing differentiability on $\mathrm{I}^\circ$ and $f, f' \in \mathrm{L}^1(\mathrm{I})$. The right-sided and left-sided $P_{cap}$ of order $\beta_o$ are described as below:*

$$ {}^{\mathrm{PC}}_{u^+}\mathrm{D}^{\beta_o}_{v} f(v) = \frac{1}{\Gamma(1-\beta_o)} \int_u^v \Big[\mathrm{K}_1(\beta_o, v-\varkappa) f(\varkappa) + \mathrm{K}_0(\beta_o, v-\varkappa) f'(\varkappa)\Big](v-\varkappa)^{-\beta_o} d\varkappa $$

*and*

$$ {}^{\mathrm{PC}}_{v^-}\mathrm{D}^{\beta_o}_{u} f(u) = \frac{1}{\Gamma(1-\beta_o)} \int_u^v \Big[\mathrm{K}_1(\beta_o, \varkappa-u) f(\varkappa) + \mathrm{K}_0(\beta_o, \varkappa-u) f'(\varkappa)\Big](\varkappa-u)^{-\beta_o} d\varkappa, $$

*where $\beta_o \in [0,1]$ and $\mathrm{K}_0(\beta_o, t) = (1-\beta_o)^2 t^{1-\beta_o}$ and $\mathrm{K}_1(\beta_o, t) = {\beta_o}^2 t^{\beta_o}$.*

### *1.2. Newton-type inequalities*

Researchers have focused a lot of emphasis on Newton-type inequalities since they have several applications in both the pure and practical sciences. As an example, integer-order Newton-type inequalities related to p-harmonic convex functions were established by Noor et al. [27]. Additionally, Sitthiwirattham et al. [35] introduced several Newton-type inequalities within the context of RL-fractional integrals. Furthermore, Luangboon et al. [20] used the (p, q)-derivatives in 2021 to derive a few Newton-type inequalities.

These days, multiplicative calculus is becoming more and more popular because of its uses in inequality theory. In [28] and [29], Ozcan studied the $H_rH_d$-type inequalities for multiplicatively P-convex functions and multiplicatively preinvex functions, respectively. Chasreechai et al. [9] created the multiplicative integral inequalities of Newton type in 2022. Additionally, several uses for the outcomes were provided. For 2023, the authors [39] derived integral inequalities of midpoint and trapezoid type by means of multiplicative functions twice differentiable. Meftah [22] deduced the multiplicative integral inequalities of the Maclaurin type in the same year. Readers will note that the topics covered in these articles included integer-order multiplicative calculus inequalities. To top it off, researchers have been tackling multiplicative fractional inequalities theory piecemeal. For example, Budak and Özçelik [6] obtained the multiplicative RL-fractional integral inequalities for the first time. Fu et al. [14] then introduced two new integral operators and analysed multiplicative fractional $H_rH_d$-type inequalities, which they called multiplicative tempered fractional integrals. Subsequently, the authors of the paper [32] proved fractional $H_rH_d$-type inequalities for *differentiable mappings by using multiplicative fractional integrals with exponential kernels.

Furthermore, the equivalent fractional $H_rH_d$-type inequalities were derived by Kashuri et al. [18] using generalised multiplicative fractional integrals. We recommend readers to the published works [12, 13, 23–25, 31, 40] for current results on fractional multiplicative calculus.

## 2. Hermite-Hadamard's Type Inequalities for Multiplicative $P-$Integrals

To initiate our article, we aim to derive the $H_rH_d$'s inequality pertaining to the Multiplicative proportional Caputo-hybrid operators.

**Theorem 2.1.** *Let $f : \mathrm{I} \subset \Re^+ \to \Re$ be differentiable function on $\mathrm{I}^\circ$, the interior of the interval $\mathrm{I}$, where $u, v \in \mathrm{I}^\circ$ with $u < v$ and $f, f^*$ be the multiplicative convex function on $\mathrm{I}$. Then the following inequalities hold:*

$$ \left[f\left(\frac{u+v}{2}\right)\right]^{{\beta_o}^2 (v-u)^{\beta_o}} . \left[f^*\left(\frac{u+v}{2}\right)\right]^{\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}} \le \left[{}_u\mathrm{D}^{\beta_o}_{*} f(v) ._{*}\mathrm{D}^{\beta_o}_{v} f(u)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}} $$
$$ \le \Big(f(u)\cdot f(v)\Big)^{2{\beta_o}^2 (v-u)^{\beta_o}} . \Big(f^*(u)\cdot f^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}. \quad (7) $$

*Proof.* Since $f$ and $f^*$ are multiplicative p-convex function on interval $[u, v]$, then we have

$$f\left(\frac{u+v}{2}\right) = f\left(\frac{ut+(1-t)v+tv+(1-t)u}{2}\right) \le [f(ut+(1-t)v)][f(tv+(1-t)u)]$$

$$\ln f\left(\frac{u+v}{2}\right) \le \ln f(tu+(1-t)v) + \ln f(tv+(1-t)u), \tag{8}$$

similarly for $f^*$

$$\ln f^*\left(\frac{u+v}{2}\right) \le \ln f^*(tu+(1-t)v) + \ln f^*(tv+(1-t)u). \tag{9}$$

Multiply (8) with ${\beta_o}^2(v-u)^{\beta_o}$ and (9) with $(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}$ respectively, we have

$$\begin{aligned} &{\beta_o}^2(v-u)^{\beta_o}\ln f\left(\frac{u+v}{2}\right) \\ &\le {\beta_o}^2(v-u)^{\beta_o}\ln f(tu+(1-t)v) + {\beta_o}^2(v-u)^{\beta_o}\ln f(tv+(1-t)u), \end{aligned} \tag{10}$$

and

$$\begin{aligned} &(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*\left(\frac{u+v}{2}\right) \\ &\le (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*(tu+(1-t)v) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*(tv+(1-t)u). \end{aligned} \tag{11}$$

Adding these two expressions by side to side, then integrate the resulting inequality with respect to $t$ over $[0, 1]$, we obtain

$$\begin{aligned} &{\beta_o}^2(v-u)^{\beta_o}\ln f\left(\frac{u+v}{2}\right) + (1-\beta_o)^2(v-u)^{1-\beta_o}\ln f^*\left(\frac{u+v}{2}\right)\int_0^1 t^{1-2\beta_o}\mathrm{dt} \\ &\le \int_0^1 \Big[{\beta_o}^2(v-u)^{\beta_o}\mathrm{t}^{\beta_o}\ln f(tu+(1-t)v) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o}\ln f^*(tu+(1-t)v)\Big]\mathrm{t}^{-\beta_o}\mathrm{dt} \\ &+ \int_0^1 \Big[{\beta_o}^2(v-u)^{\beta_o}\mathrm{t}^{\beta_o}\ln f(tv+(1-t)u) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o}\ln f^*(tv+(1-t)u)\Big]\mathrm{t}^{-\beta_o}\mathrm{dt}. \end{aligned}$$

Using the change of variable, we obtain

$$\begin{aligned} &{\beta_o}^2(v-u)^{\beta_o}\ln f\left(\frac{u+v}{2}\right) + \frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}\ln f^*\left(\frac{u+v}{2}\right) \\ &\le \frac{1}{(v-u)^{1-\beta_o}}\int_u^v \Big[{\beta_o}^2(v-\varkappa)^{\beta_o}\ln f(\varkappa) + (1-\beta_o)^2(v-\varkappa)^{1-\beta_o}\ln f^*(\varkappa)\Big](v-\varkappa)^{-\beta_o}\mathrm{d}\varkappa \\ &+ \frac{1}{(v-u)^{1-\beta_o}}\int_u^v \Big[{\beta_o}^2(\varkappa-u)^{\beta_o}\ln f(\varkappa) + (1-\beta_o)^2(\varkappa-u)^{1-\beta_o}\ln f^*(\varkappa)\Big](\varkappa-u)^{-\beta_o}\mathrm{d}\varkappa \\ &= \frac{1}{(v-u)^{1-\beta_o}}\int_u^v \Big[\mathrm{K}_1\big(\beta_o, (v-\varkappa)\big)\ln f(\varkappa) + \mathrm{K}_0\big(\beta_o, (v-\varkappa)\big)\ln f^*(\varkappa)\Big](v-\varkappa)^{-\beta_o}\mathrm{d}\varkappa \end{aligned}$$

$$+\frac{1}{(v-u)^{1-\beta_o}}\int_u^v\Big[\mathrm{K}_1\Big(\beta_o,(\varkappa-u)\Big)\ln f(\varkappa)+\mathrm{K}_0\Big(\beta_o,(\varkappa-u)\Big)\ln \mathrm{f}^*(\varkappa)\Big](\varkappa-u)^{-\beta_o}\mathrm{d}\varkappa$$
$$=\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}\Big[{}^{\mathrm{PC}}{}_{u^+}\mathrm{D}_v^{\beta_o}\ln f(v)+{}^{\mathrm{PC}}{}_{v^-}\mathrm{D}_u^{\beta_o}\ln f(u)\Big].$$

Thus we get,

$$\Big[f\Big(\frac{u+v}{2}\Big)\Big]^{\beta_o{}^2(v-u)^{\beta_o}}.\Big[\mathrm{f}^*\Big(\frac{u+v}{2}\Big)\Big]^{\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}}$$
$$\leq\Big[\exp\Big\{{}^{\mathrm{PC}}{}_u^+\mathrm{D}_v^{\beta_o}\ln f(v)+{}^{\mathrm{PC}}{}_v^-\mathrm{D}_u^{\beta_o}\ln f(u)\Big\}\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$
$$=\Big[{}_u\mathrm{D}_*^{\beta_o}f(v)._*\mathrm{D}_v^{\beta_o}f(u)\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$

This is the first part of inequality (14).
As $f$ and $\mathrm{f}^*$ are Multiplicative convex functions on $t\in[0,1]$, then we have

$$f(ut+(1-t)v)\leq f(u)f(v)$$

$$f(vt+(1-t)u)\leq f(v)f(u)$$

i.e.

$$\begin{aligned}&\ln f(ut+(1-t)v)+\ln f(vt+(1-t)u)\\&\leq\ln f(u)+\ln f(v)+\ln f(u)+\ln f(v)\\&=2\ln f(u)+2\ln f(v)\end{aligned}\tag{12}$$

silimarly for $\mathrm{f}^*$

$$\begin{aligned}&\ln \mathrm{f}^*(ut+(1-t)v)+\ln \mathrm{f}^*(vt+(1-t)u)\\&\leq 2\ln \mathrm{f}^*(u)+2\ln \mathrm{f}^*(v)\end{aligned}\tag{13}$$

Multiply (12) with $\beta_o{}^2(v-u)^{\beta_o}$ and (13) with $(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}$, we have

$$\begin{aligned}&\beta_o{}^2(v-u)^{\beta_o}\ln f(ut+(1-t)v)+\beta_o{}^2(v-u)^{\beta_o}\ln f(vt+(1-t)u)\\&\qquad\leq\beta_o{}^2(v-u)^{\beta_o}[2\ln f(u)+2\ln f(v)]\end{aligned}$$

and

$$\begin{aligned}&(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln \mathrm{f}^*(ut+(1-t)v)\\&+(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln \mathrm{f}^*(vt+(1-t)u)\\&\leq(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}[2\ln \mathrm{f}^*(u)+2\ln \mathrm{f}^*(v)]\end{aligned}$$

Adding these two expressions by side to side, then integrate the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\int_0^1\Big[\beta_o{}^2(v-u)^{\beta_o}t^{\beta_o}\ln f(ut+(1-t)v)$$
$$+(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o}\ln \mathrm{f}^*(ut+(1-t)v)\Big]t^{-\beta_o}\mathrm{d}t$$

$$+\int_0^1 \Big[\beta_o{}^2(v-u)^{\beta_o} t^{\beta_o} \ln f(vt+(1-t)u)$$
$$+(1-\beta_o)^2(v-u)^{1-\beta_o} t^{1-\beta_o} \ln f^*(vt+(1-t)u)\Big] t^{-\beta_o} dt$$
$$\leq \beta_o{}^2(v-u)^{\beta_o}\Big[2\ln f(u)+2\ln f(v)\Big]+(1-\beta_o)(v-u)^{1-\beta_o}\Big[\ln f^*(u)+\ln f^*(v)\Big].$$

Using the change of the variable, we obtain that

$$\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}\Big[{}^{PC}{}_{u^+}D_v^{\beta_o}\ln f(v)+{}^{PC}{}_{v^-}D_u^{\beta_o}\ln f(u)\Big]$$
$$\leq \beta_o{}^2(v-u)^{\beta_o}\Big[2\ln f(u)+2\ln f(v)\Big]+(1-\beta_o)(v-u)^{1-\beta_o}\Big[\ln f^*(u)+\ln f^*(v)\Big]$$

$$\Big[{}_uD_*^{\beta_o} f(v)._*D_v^{\beta_o} f(u)\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$
$$\leq \Big(f(u)\cdot f(v)\Big)^{2\beta_o{}^2(v-u)^{\beta_o}}.\Big(f^*(u)\cdot f^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}.$$

Which completes the proof. □

**Example 2.2.** *The Figure 2.1 describes the validity of the inequalities of Theorem 2.1 for $f(x)=3^x$, $u=0$, $v=1$ and $\beta_o\in[0,1]$.*

**Corollary 2.3.** *If $f$ and g are two positive and multiplicative convex functions,then we have the following inequality*

$$\Big[f\Big(\frac{u+v}{2}\Big)\cdot g\Big(\frac{u+v}{2}\Big)\Big]^{\beta_o{}^2(v-u)^{\beta_o}}\cdot\Big[f^*\Big(\frac{u+v}{2}\Big)\cdot g^*\Big(\frac{u+v}{2}\Big)\Big]^{\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}}$$
$$\leq \Big[{}_uD_*^{\beta_o} fg(v)._*D_v^{\beta_o} fg(u)\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$
$$\leq \Big(f(u)f(v)\Big)^{2\beta_o{}^2(v-u)^{\beta_o}}.\Big(f^*(u)f^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}\cdot\Big(g(u)g(v)\Big)^{2\beta_o{}^2(v-u)^{\beta_o}}.\Big(g^*(u)g^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}.$$

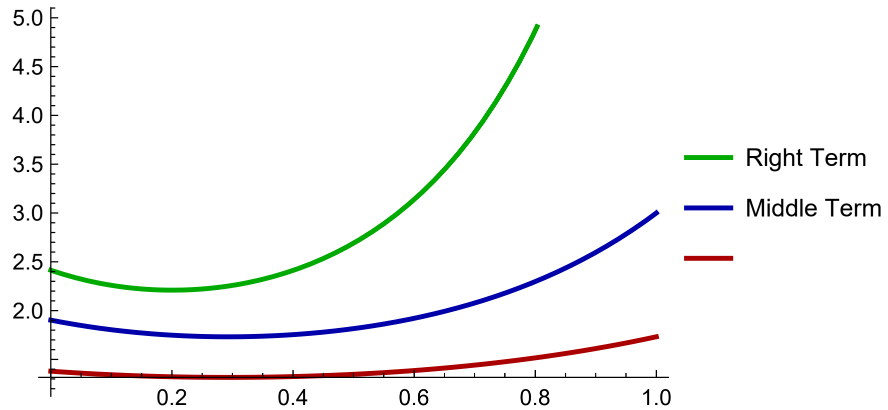
*Proof.* *Since $f$ and g are positive and multiplicative p-convex functions, then fg is positive and multiplicative p-convex function. Thus, if we apply Theorem 2.1 to the function fg,then we obtain the corollary 28.* □

**Remark 2.4.** *If we take $\beta_o=1$ in corollary 28, then we have the following inequality*

$$f\Big(\frac{u+v}{2}\Big)\cdot g\Big(\frac{u+v}{2}\Big)\leq\Big(\int_u^v (f(\varkappa))^{d\varkappa}\cdot\int_u^v (g(\varkappa))^{d\varkappa}\Big)^{\frac{1}{v-u}}$$
$$\leq (f(u)f(v))^2\cdot(g(u)g(v))^2.$$

**Theorem 2.5.** *Let $f:\mathrm{I}\subset\mathfrak{R}^+\to\mathfrak{R}$ be differentiable function on $\mathrm{I}^\circ$, the interior of the interval I, where $u,v\in\mathrm{I}^\circ$ with $u<v$ and $f,f^*$ be the multiplicative $p-$convex function on I. Then the following inequalities hold:*

$$\Big[f\Big(\frac{u+v}{2}\Big)\Big]^{\beta_o{}^2(v-u)^{\beta_o}}.\Big[f^*\Big(\frac{u+v}{2}\Big)\Big]^{2^{2\beta_o-2}(1-\beta_o)(v-u)^{1-\beta_o}}$$
$$\leq\Big[{}_{\frac{u+v}{2}}D_*^{\beta_o} f(v)._*D_{\frac{u+v}{2}}^{\beta_o} f(u)\Big]^{2\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$
$$\leq\Big(f(u)f(v)\Big)^{2\beta_o{}^2(v-u)^{\beta_o}}.\Big(f^*(u)\cdot f^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}. \tag{14}$$




(a) *Graphical and Numerical illustration of Theorem* 2.1.

| Values of $\beta_o$ | Left Term | Middle Term | Right Term |
|---|---|---|---|
| 0 | 1.37944 | 1.90285 | 2.4139 |
| 0.1 | 1.34313 | 1.804 | 2.25938 |
| 0.2 | 1.32222 | 1.74827 | 2.20976 |
| 0.3 | 1.31602 | 1.73191 | 2.25832 |
| 0.4 | 1.32431 | 1.75381 | 2.41163 |
| 0.5 | 1.34738 | 1.81544 | 2.69104 |
| 0.6 | 1.386 | 1.92099 | 3.13772 |
| 0.7 | 1.44147 | 2.07783 | 3.82291 |
| 0.8 | 1.51572 | 2.29741 | 4.86697 |
| 0.9 | 1.6114 | 2.59662 | 6.47453 |
| 1 | 1.73205 | 3 | 9 |

(b) *Numerical illustration of Theorem* 2.1.

Figure 2.1: Graphical and Numerical illustration of Theorem 2.1.

*Proof.* Since $f$ and $f^*$ are multiplicative p-convex function on interval $[u, v]$, then we have

$$f\left(\frac{u+v}{2}\right) = f\left(\frac{u\frac{t}{2} + \left(\frac{2-t}{2}\right)v + \frac{t}{2}v + \left(\frac{2-t}{2}\right)u}{2}\right)$$

$$\leq \left[f(u\frac{t}{2} + \left(\frac{2-t}{2}\right)v)\right]\left[f(v\frac{t}{2} + \left(\frac{2-t}{2}\right)u)\right]$$

$$\ln f\left(\frac{u+v}{2}\right) \leq \ln f\left(\frac{t}{2}u + \left(\frac{2-t}{2}\right)v\right) + \ln f\left(\frac{t}{2}v + \left(\frac{2-t}{2}\right)u\right), \tag{15}$$

similarly for $f^*$

$$\ln f^*\left(\frac{u+v}{2}\right) \leq \ln f^*\left(\frac{t}{2}u + \left(\frac{2-t}{2}\right)v\right) + \ln f^*\left(\frac{t}{2}v + \left(\frac{2-t}{2}\right)u\right). \tag{16}$$

Multiply (15) with $\beta_o{}^2(v-u)^{\beta_o}$ and (16) with $2^{2\beta_o-1}(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}$ respectively, we have

$$\beta_o{}^2(v-u)^{\beta_o}\ln f\left(\frac{u+v}{2}\right)$$

$$\leq \beta_o{}^2(v-u)^{\beta_o}\ln f\left(\frac{t}{2}u + \left(\frac{2-t}{2}\right)v\right) + \beta_o{}^2(v-u)^{\beta_o}\ln f\left(\frac{t}{2}v + \left(\frac{2-t}{2}\right)u\right), \tag{17}$$

and

$$
\begin{aligned}
&(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*\Big(\frac{u+v}{2}\Big)\\
&\leq (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*\Big(\frac{t}{2}u+\Big(\frac{2-t}{2}\Big)v\Big)\\
&+(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}\ln f^*\Big(\frac{t}{2}v+\Big(\frac{2-t}{2}\Big)u\Big). \qquad (18)
\end{aligned}
$$

Adding these two expressions by side to side, then integrate the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$
\begin{aligned}
&\beta_o{}^2(v-u)^{\beta_o}\ln f\Big(\frac{u+v}{2}\Big)+(1-\beta_o)^2(v-u)^{1-\beta_o}\ln f^*\Big(\frac{u+v}{2}\Big)\int_0^1 t^{1-2\beta_o}\mathrm{dt}\\
&\leq \int_0^1\Big[\beta_o{}^2(v-u)^{\beta_o}t^{\beta_o}\ln f\Big(\frac{t}{2}u+(1-\frac{t}{2})v\Big)\\
&+(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o}\ln f^*\Big(\frac{t}{2}u+(1-\frac{t}{2})v\Big)\Big]t^{-\beta_o}\mathrm{d}t\\
&+\int_0^1\Big[\beta_o{}^2(v-u)^{\beta_o}t^{\beta_o}\ln f\Big(\frac{t}{2}v+(1-\frac{t}{2})u\Big)\\
&+(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o}\ln f^*\Big(\frac{t}{2}v+(1-\frac{t}{2})u\Big)\Big]t^{-\beta_o}\mathrm{d}t.
\end{aligned}
$$

Using the change of variable, we obtain

$$
\begin{aligned}
&\beta_o{}^2(v-u)^{\beta_o}\ln f\Big(\frac{u+v}{2}\Big)+\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}\ln f^*\Big(\frac{u+v}{2}\Big)\\
&\leq \frac{1}{(v-u)^{1-\beta_o}}\int_u^v\Big[\beta_o{}^2(v-\varkappa)^{\beta_o}\ln f(\varkappa)+(1-\beta_o)^2(v-\varkappa)^{1-\beta_o}\ln f^*(\varkappa)\Big](v-\varkappa)^{-\beta_o}\mathrm{d}\varkappa\\
&+\frac{1}{(v-u)^{1-\beta_o}}\int_u^v\Big[\beta_o{}^2(\varkappa-u)^{\beta_o}\ln f(\varkappa)+(1-\beta_o)^2(\varkappa-u)^{1-\beta_o}\ln f^*(\varkappa)\Big](\varkappa-u)^{-\beta_o}\mathrm{d}\varkappa\\
&=\frac{1}{(v-u)^{1-\beta_o}}\int_u^v\Big[\mathrm{K}_1\big(\beta_o,(v-\varkappa)\big)\ln f(\varkappa)+\mathrm{K}_0\big(\beta_o,(v-\varkappa)\big)\ln f^*(\varkappa)\Big](v-\varkappa)^{-\beta_o}\mathrm{d}\varkappa\\
&+\frac{1}{(v-u)^{1-\beta_o}}\int_u^v\Big[\mathrm{K}_1\big(\beta_o,(\varkappa-u)\big)\ln f(\varkappa)+\mathrm{K}_0\big(\beta_o,(\varkappa-u)\big)\ln f^*(\varkappa)\Big](\varkappa-u)^{-\beta_o}\mathrm{d}\varkappa\\
&=\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}\Big[{}^{\mathrm{PC}}{}_{u^+}\mathrm{D}_v^{\beta_o}\ln f(v)+{}^{\mathrm{PC}}{}_{v^-}\mathrm{D}_u^{\beta_o}\ln f(u)\Big].
\end{aligned}
$$

Thus we get,

$$
\begin{aligned}
&\Big[f\Big(\frac{u+v}{2}\Big)\Big]^{\beta_o{}^2(v-u)^{\beta_o}}.\Big[f^*\Big(\frac{u+v}{2}\Big)\Big]^{\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}}\\
&\leq\Big[\exp\Big\{{}^{\mathrm{PC}}{}^{+}_{u}\mathrm{D}_v^{\beta_o}\ln f(v)+{}^{\mathrm{PC}}{}^{-}_{v}\mathrm{D}_u^{\beta_o}\ln f(u)\Big\}\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}\\
&=\Big[{}_u\mathrm{D}_*^{\beta_o}f(v)._*\mathrm{D}_v^{\beta_o}f(u)\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}
\end{aligned}
$$

This is the first part of inequality (14).
As $f$ and $f^*$ are Multiplicative convex functions on $t\in[0,1]$, then we have

$$f(ut+(1-t)v)\leq f(u)f(v)$$

$$f(vt+(1-t)u) \le f(v)f(u)$$

i.e.

$$\begin{aligned} &\ln f(ut+(1-t)v) + \ln f(vt+(1-t)u) \\ &\le \ln f(u) + \ln f(v) + \ln f(u) + \ln f(v) \\ &= 2\ln f(u) + 2\ln f(v) \end{aligned} \tag{19}$$

silimarly for f*

$$\begin{aligned} &\ln f^*(ut+(1-t)v) + \ln f^*(vt+(1-t)u) \\ &\le 2\ln f^*(u) + 2\ln f^*(v) \end{aligned} \tag{20}$$

Multiply (19) with ${\beta_o}^2(v-u)^{\beta_o}$ and (20) with $(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}$, we have

$$\begin{aligned} {\beta_o}^2(v-u)^{\beta_o} \ln f(ut+(1-t)v) + {\beta_o}^2(v-u)^{\beta_o} \ln f(vt+(1-t)u) \\ \le {\beta_o}^2(v-u)^{\beta_o}[2\ln f(u) + 2\ln f(v)] \end{aligned}$$

and

$$\begin{aligned} &(1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o} \ln f^*(ut+(1-t)v) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o} \ln f^*(vt+(1-t)u) \\ &\le (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-2\beta_o}[2\ln f^*(u) + 2\ln f^*(v)] \end{aligned}$$

Adding these two expressions by side to side, then integrate the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 \Big[{\beta_o}^2(v-u)^{\beta_o}t^{\beta_o} \ln f(ut+(1-t)v) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o} \ln f^*(ut+(1-t)v)\Big]t^{-\beta_o}\,\mathrm{d}t \\ &+ \int_0^1 \Big[{\beta_o}^2(v-u)^{\beta_o}t^{\beta_o} \ln f(vt+(1-t)u) \\ &+ (1-\beta_o)^2(v-u)^{1-\beta_o}t^{1-\beta_o} \ln f^*(vt+(1-t)u)\Big]t^{-\beta_o}\,\mathrm{d}t \\ &\le {\beta_o}^2(v-u)^{\beta_o}\Big[2\ln f(u) + 2\ln f(v)\Big] + (1-\beta_o)(v-u)^{1-\beta_o}\Big[\ln f^*(u) + \ln f^*(v)\Big]. \end{aligned}$$

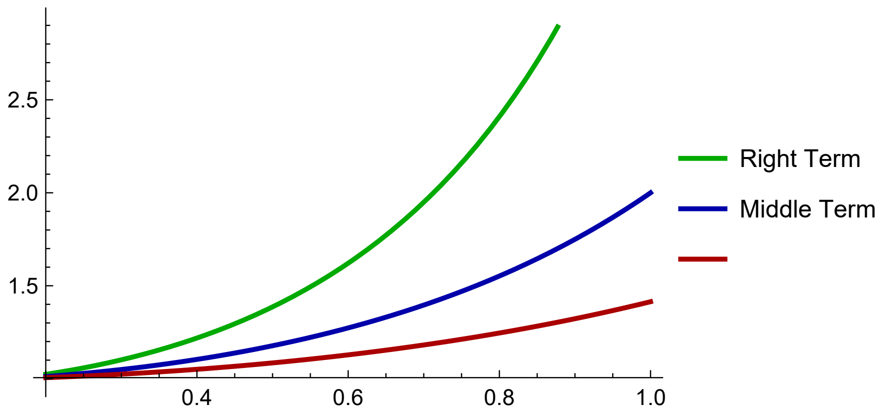
Using the change of the variable, we obtain that

$$\begin{aligned} &\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}\Big[{}^{PC}{}_{u^+}\mathrm{D}_v^{\beta_o} \ln f(v) + {}^{PC}{}_{v^-}\mathrm{D}_u^{\beta_o} \ln f(u)\Big] \\ &\le {\beta_o}^2(v-u)^{\beta_o}\Big[2\ln f(u) + 2\ln f(v)\Big] + (1-\beta_o)(v-u)^{1-\beta_o}\Big[\ln f^*(u) + \ln f^*(v)\Big] \end{aligned}$$

$$\begin{aligned} &\Big[{}_u\mathrm{D}_*^{\beta_o} f(v) \cdot {}_*\mathrm{D}_v^{\beta_o} f(u)\Big]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}} \\ &\le \Big(f(u)\cdot f(v)\Big)^{2{\beta_o}^2(v-u)^{\beta_o}} . \Big(f^*(u)\cdot f^*(v)\Big)^{(1-\beta_o)(v-u)^{1-\beta_o}}. \end{aligned}$$

Which completes the proof. □

**Example 2.6.** *The following figure 2.2 describe the validity of the inequalities of Theorem 2.5 for* $f(x) = 2^x$, $u = 0$, $v = 1$ *and* $\beta_o \in [0, 1]$.




(a) *Graphical illustration of Theorem* 2.5.

| Values of $\beta_o$ | Left Term | Middle Term | Right Term |
|---|---|---|---|
| 0.2 | 1.0059 | 1.01184 | 1.02383 |
| 0.3 | 1.02451 | 1.04962 | 1.1017 |
| 0.4 | 1.05071 | 1.104 | 1.21882 |
| 0.5 | 1.08509 | 1.17741 | 1.38629 |
| 0.6 | 1.12838 | 1.27323 | 1.62112 |
| 0.7 | 1.18155 | 1.39607 | 1.94901 |
| 0.8 | 1.24584 | 1.55213 | 2.4091 |
| 0.9 | 1.32277 | 1.74972 | 3.06152 |
| 1 | 1.41421 | 2 | 4 |

(b) *Numerical illustration of Theorem* 2.5.

Figure 2.2: Graphical and Numerical illustration of Theorem 2.5.

**Corollary 2.7.** *If f and g are two positive and multiplicative convex functions,then we have the following inequality*

$$\left[f\left(\frac{u+v}{2}\right)\cdot g\left(\frac{u+v}{2}\right)\right]^{\beta_o{}^2(v-u)^{\beta_o}} \cdot \left[f^*\left(\frac{u+v}{2}\right)\cdot g^*\left(\frac{u+v}{2}\right)\right]^{\frac{1}{2}(1-\beta_o)(v-u)^{1-\beta_o}}$$
$$\leq \left[{}_uD_*^{\beta_o} fg(v) ._* D_v^{\beta_o} fg(u)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{1-\beta_o}}}$$
$$\leq \left(f(u)f(v)\right)^{2\beta_o{}^2(v-u)^{\beta_o}} . \left(f^*(u)f^*(v)\right)^{(1-\beta_o)(v-u)^{1-\beta_o}}$$
$$\cdot \left(g(u)g(v)\right)^{2\beta_o{}^2(v-u)^{\beta_o}} . \left(g^*(u)g^*(v)\right)^{(1-\beta_o)(v-u)^{1-\beta_o}}.$$

*Proof.* *Since f and g are positive and multiplicative p-convex functions, then* fg *is positive and multiplicative p-convex function. Thus, if we apply Theorem 2.1 to the function* fg*,then we obtain the corollary 28.* □

**Remark 2.8.** *If we take* $\beta_o = 1$ *in corollary 28, then we have the following inequality*

$$f\left(\frac{u+v}{2}\right)\cdot g\left(\frac{u+v}{2}\right) \leq \left(\int_u^v (f(\varkappa))^{d\varkappa} \cdot \int_u^v (g(\varkappa))^{d\varkappa}\right)^{\frac{1}{v-u}}$$
$$\leq (f(u)f(v))^2 \cdot (g(u)g(v))^2.$$

## 3. Newton's Type Inequalities for Multiplicative p-Integrals

To prove our other main results, we require the following lemma.

**Lemma 3.1.** *Let $f : \mathrm{I} \subset \mathfrak{R}^{+} \to \mathfrak{R}$ be a multiplicative differentiable function on* $\mathrm{I}^{\circ}$*, the interior of the interval I,where* $u, v \in \mathrm{I}^{\circ}$ *with* $u < v$ *and* $f^{*}, f^{**} \in L[u, v]$*. Then the following identity hold*

$$\left[f(u)\, f\Big(\frac{2u+v}{3}\Big)^{3} f\Big(\frac{u+2v}{3}\Big)^{3} f(v)\right]^{\frac{\beta_o{}^2}{8}(v-u)^{\beta_o-1}}$$
$$\cdot\left[\mathrm{f}^{*}(u)\, \mathrm{f}^{*}\left(\frac{2u+v}{3}\right)^{3} \mathrm{f}^{*}\left(\frac{u+2v}{3}\right)^{3} \mathrm{f}^{*}(v)\right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{16.3^{1-2\beta_o}}}$$
$$\cdot\frac{1}{\left[{}_{*}\mathrm{D}^{\beta_o}_{\left(\frac{2u+v}{3}\right)} f(u)\cdot {}_{*}\mathrm{D}^{\beta_o}_{\left(\frac{u+2v}{3}\right)} f\Big(\frac{2u+v}{3}\Big)\cdot {}_{*}\mathrm{D}^{\beta_o}_{(v)} f\Big(\frac{u+2v}{3}\Big)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{\beta_o-2}}}}$$
$$= \mathfrak{I}_1 \times \mathfrak{I}_2 \times \mathfrak{I}_3 \times \mathfrak{I}_4 \times \mathfrak{I}_5 \times \mathfrak{I}_6,$$

*where*

$$\mathfrak{I}_1 = \left(\int_0^1 \left(\mathrm{f}^{*}\left((1-t)\, u + t\frac{2u+v}{3}\right)^{\left(t-\frac{3}{8}\right)}\right)^{dt}\right)^{\frac{\beta_o{}^2(v-u)^{\beta_o}}{9}},$$

$$\mathfrak{I}_2 = \left(\left(\int_0^1 \left(\mathrm{f}^{**}\left((1-t)\, u\right) + t\frac{2u+v}{3}\right)^{\left(t^{2-2\beta_o}-\frac{3}{8}\right)}\right)^{dt}\right)^{\frac{(1-\beta_o)(v-u)^{3-\beta_o}}{2.3^{3-2\beta_o}}},$$

$$\mathfrak{I}_3 = \left(\int_0^1 \left(\mathrm{f}^{*}\left((1-t)\, \frac{2u+v}{3} + t\frac{u+2v}{3}\right)^{\left(t-\frac{1}{2}\right)}\right)^{dt}\right)^{\frac{\beta_o{}^2(v-u)^{\beta_o}}{9}},$$

$$\mathfrak{I}_4 = \left(\left(\int_0^1 \left(\mathrm{f}^{**}\left((1-t)\, \frac{2u+v}{3}\right) + t\frac{u+2v}{3}\right)^{\left(t^{2-2\beta_o}-\frac{1}{2}\right)}\right)^{dt}\right)^{\frac{(1-\beta_o)(v-u)^{3-\beta_o}}{2.3^{3-2\beta_o}}}$$

$$,\mathfrak{I}_5 = \left(\int_0^1 \left(\mathrm{f}^{*}\left((1-t)\, \frac{u+2v}{3} + tv\right)^{\left(t-\frac{5}{8}\right)}\right)^{dt}\right)^{\frac{\beta_o{}^2(v-u)^{\beta_o}}{9}}$$

*and*

$$\mathfrak{I}_6 = \left(\left(\int_0^1 \left(\mathrm{f}^{**}\left((1-t)\, \frac{u+2v}{3}\right) + tv\right)^{\left(t^{2-2\beta_o}-\frac{5}{8}\right)}\right)^{dt}\right)^{\frac{(1-\beta_o)(v-u)^{3-\beta_o}}{2.3^{3-2\beta_o}}}.$$

*Proof.* Using integration by parts for multiplicative integrals from $\mathfrak{I}_1$, we have

$$\mathfrak{I}_1 = \left(\int_0^1 \left(\mathrm{f}^{*}\left((1-t)\, u + t\frac{2u+v}{3}\right)^{\left(t-\frac{3}{8}\right)}\right)^{dt}\right)^{\frac{\beta_o{}^2(v-u)^{\beta_o}}{9}}$$

$$= \int_0^1 \left( f^* \left( (1-t)\, u + t \frac{2u+v}{3} \right)^{\frac{(v-u)}{3} \left( \frac{\beta_o{}^2}{3} (v-u)^{\beta_o - 1} \right) \left( t - \frac{3}{8} \right)} \right)^{dt}$$

$$= \frac{f\left(\frac{2u+v}{2}\right)^{\frac{5}{24}\beta_o{}^2 (v-u)^{\beta_o-1}}}{f(u)^{\beta_o{}^2 (v-u)^{\beta_o-1} \left(\frac{-1}{8}\right)}} \cdot \frac{1}{\int_0^1 \left( f\left( (1-t)\, u + t\frac{2u+v}{3} \right)^{\frac{\beta_o{}^2 (v-u)^{\beta_o-1}}{3}} \right)^{dt}}$$

$$= \left[ f\left( \frac{2u+v}{2} \right)^{\frac{5}{24}} \cdot f(u)^{\frac{1}{8}} \right]^{\beta_o{}^2 (v-u)^{\beta_o-1}}$$

$$\cdot \frac{1}{exp\left\{ \frac{\beta_o{}^2 (v-u)^{\beta_o-1}}{3} \int_0^1 \ln f\left( (1-t)\, u + t \frac{2u+v}{3} \right) dt \right\}}$$

$$= \left[ f\left( \frac{2u+v}{2} \right)^{\frac{5}{24}} \cdot f(u)^{\frac{1}{8}} \right]^{\beta_o{}^2 (v-u)^{\beta_o-1}} \cdot \frac{1}{exp\left\{ \frac{1}{(v-u)^{2-\beta_o}} \int_u^{\frac{2u+v}{3}} \beta_o{}^2 \ln f(\varkappa)\, d\varkappa \right\}} \tag{21}$$

Similarly for $\Im_2$

$$\Im_2 = \int_0^1 \left( f^{**} \left( (1-t)\, u + t \frac{2u+v}{3} \right)^{\frac{v-u}{3} \left( \frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{2-2\beta_o}} \right) \left( t^{2-2\beta_o} - \frac{3}{8} \right)} \right)^{dt}$$

$$= \left[ f^* \left( \frac{2u+v}{2} \right)^{\frac{5}{24}} \cdot f^*(u)^{\frac{1}{8}} \right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}}$$

$$\cdot \frac{1}{\int_0^1 \left( f\left( (1-t)\, u + t\frac{2u+v}{3} \right)^{\frac{(1-\beta_o)^2 (v-u)^{2-\beta_o}}{3^{2-2\beta_o}}} \mu_0{}^{1-2\beta_o} \right)^{dt}},$$

$$= \left[ f^* \left( \frac{2u+v}{2} \right)^{\frac{5}{24}} \cdot f^*(u)^{\frac{1}{8}} \right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}}$$

$$\cdot \frac{1}{exp\left\{ \frac{(1-\beta_o)^2 (v-u)^{2-\beta_o}}{3^{2-2\beta_o}} \int_0^1 \ln\left( f\left( (1-t)\, u + t\frac{2u+v}{3} \right) \right) \mu_0{}^{1-2\beta_o} dt \right\}},$$

$$= \left[ f^* \left( \frac{2u+v}{2} \right)^{\frac{5}{24}} f^*(u)^{\frac{1}{8}} \right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}}$$

$$\cdot \frac{1}{exp\left\{ \frac{1}{(v-u)^{2-\beta_o}} \int_a^{\frac{2u+v}{3}} (\varkappa - u)^{1-2\beta_o} (1-\beta_o)^2 \ln f^*(\varkappa)\, d\varkappa \right\}} \tag{22}$$

Using (21) and (22), we get

$$\Im_1 \times \Im_2 = \frac{\left[f\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f(u)^{\frac{1}{8}}\right]^{\beta_o^2 (v-u)^{\beta_o - 1}} \cdot \left[f^*\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f^*(u)^{\frac{1}{8}}\right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}}}{\exp\left\{\frac{1}{(v-u)^{2-\beta_o}} \int_a^{\frac{2u+v}{3}} \left({\beta_o}^2 (\varkappa - u)^{\beta_o} \ln f(\varkappa) + (1-\beta_o)^2 (\varkappa - u)^{1-\beta_o} f^*(\varkappa)\right)(\varkappa - u)^{-\beta_o} d\varkappa\right\}}$$

$$= \left[f\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f(u)^{\frac{1}{8}}\right]^{\beta_o^2 (v-u)^{\beta_o - 1}}$$

$$\cdot \left[f^*\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f^*(u)^{\frac{1}{8}}\right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}} \cdot \frac{1}{\left[{}_*D^{\beta_o}_{\frac{2u+v}{3}} f(u)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{2-\beta_o}}}}. \tag{23}$$

By using similar method

$$\Im_3 \cdot \Im_4 = \left[f\left(\frac{u+2v}{3}\right)^{\frac{1}{6}} \cdot f\left(\frac{2u+v}{3}\right)^{\frac{1}{6}}\right]^{\beta_o^2 (v-u)^{\beta_o - 1}}$$

$$\cdot \left[f^*\left(\frac{u+2v}{3}\right)^{\frac{1}{6}} \cdot f^*\left(\frac{2u+v}{3}\right)^{\frac{1}{6}}\right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}} \cdot \frac{1}{\left[{}_*D^{\beta_o}_{\frac{u+2v}{3}} f\left(\frac{2u+2v}{3}\right)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{2-\beta_o}}}}, \tag{24}$$

and

$$\Im_5 \cdot \Im_6 = \left[f\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f(u)^{\frac{1}{8}}\right]^{\beta_o^2 (v-u)^{\beta_o - 1}}$$

$$\cdot \left[f^*\left(\frac{2u+v}{2}\right)^{\frac{5}{24}} \cdot f^*(u)^{\frac{1}{8}}\right]^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{1-2\beta_o}}} \cdot \frac{1}{\left[{}_*D^{\beta_o}_{v} f\left(\frac{2u+v}{3}\right)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{2-\beta_o}}}}. \tag{25}$$

Using (23), (24) and (25), we get the required result. □

**Remark 3.2.** *Put $\beta_o = 1$ in lemma 3.1, we get Remark 4.7 of [41].*

$$\left[f(u)\, f\left(\frac{2u+v}{3}\right)^3 f\left(\frac{u+2v}{3}\right)^3 f(v)\right]^{\frac{1}{8}} \left(\int_a^b f(\varkappa)^{d\varkappa}\right)^{\frac{1}{u-v}}$$

$$= \left(\int_0^1 \left(f^*\left((1-t)\, u + t\frac{2u+v}{3}\right)^{\left(t-\frac{3}{8}\right)}\right)^{dt}\right)^{\frac{v-u}{9}}$$

$$\cdot \left(\int_0^1 \left(f^*\left((1-t)\, \frac{2u+v}{3} + t\frac{u+2v}{3}\right)^{\left(t-\frac{1}{2}\right)}\right)^{dt}\right)^{\frac{v-u}{9}}$$

$$\cdot \left(\int_0^1 \left(f^*\left((1-t)\, \frac{u+2v}{3} + tv\right)^{\left(t-\frac{5}{8}\right)}\right)^{dt}\right)^{\frac{v-u}{9}}.$$

Next, we will consider

$$\mathfrak{M}_{\mathfrak{f}}(u,v)=\Bigg|\left[f(u)\,f\Big(\frac{2u+v}{3}\Big)^3 f\Big(\frac{u+2v}{3}\Big)^3 f(v)\right]^{{\beta_o}^2(v-u)^{\beta_o}}$$
$$\cdot\left[\mathrm{f}^*(u)\,\mathrm{f}^*\Big(\frac{2u+v}{3}\Big)^3 \mathrm{f}^*\Big(\frac{u+2v}{3}\Big)^3 \mathrm{f}^*(v)\right]^{\frac{(1-\beta_o)(v-u)^{1-\beta_o}}{2.3^{1-2\beta_o}}}$$
$$\cdot\frac{1}{\left[{}_*\mathrm{D}^{\beta_o}_{\left(\frac{2u+v}{3}\right)}f(u)\cdot{}_*\mathrm{D}^{\beta_o}_{\left(\frac{u+2v}{3}\right)}f\Big(\frac{2u+v}{3}\Big)\cdot{}_*\mathrm{D}^{\beta_o}_{(v)}f\Big(\frac{u+2v}{3}\Big)\right]^{\frac{\Gamma(1-\beta_o)}{(v-u)^{\beta_o-1}}}}\Bigg|,$$

**Theorem 3.3.** *Let $f: I\subset\mathfrak{R}^+\to\mathfrak{R}$ be a multiplicative differentiable mapping on $[u,v]$ with $u<v$ and $f^*, f^{**}$ is multiplicative convex on $[u,v]$. Then the following identity hold*

$$\mathfrak{M}_{\mathfrak{f}}(u,v)\le(\mathrm{f}^*(u)(\mathrm{f}^*(v)))^{\frac{25{\beta_o}^2(v-u)^{1+\beta_o}}{228}}$$
$$\cdot(\mathrm{f}^{**}(u)(\mathrm{f}^{**}(v)))^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left[\mathfrak{A}_g(\beta_o)+\mathfrak{A}(\beta_o)+\mathfrak{A}(\beta_o)\right]},$$

*where*

$$\mathfrak{A}_g(\beta_o)=\frac{1}{3-2\beta_o}\left[1-2\Big(\frac{3}{8}\Big)^{3-2\beta_o}\right]-\frac{3}{32},$$
$$\mathfrak{A}(\beta_o)=\frac{1}{3-2\beta_o}\left[1-\Big(\frac{1}{2}\Big)^{2-2\beta_o}\right],$$

*and*

$$\mathfrak{A}(\beta_o)=\frac{1}{3-2\beta_o}\left[1-2\Big(\frac{5}{8}\Big)^{3-2\beta_o}\right]+\frac{5}{32}.$$

*Proof.* Taking modulus in Lemma 3.1, we get

$$\mathfrak{M}_{\mathfrak{f}}(u,v)$$
$$\le\left|exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\int_0^1\Big(t-\frac{3}{8}\Big)\ln\Big(\mathrm{f}^*\Big((1-t)\,u+t\frac{2u+v}{3}\Big)\Big)dt\right\}\right|$$
$$\times\left|exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\right.\right.$$
$$\left.\left.\int_0^1\Big(t^{2-2\beta_o}-\frac{3}{8}\Big)\ln\Big(\mathrm{f}^{**}\Big((1-t)\,u+t\frac{2u+v}{3}\Big)\Big)dt\right\}\right|$$
$$\times\left|exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\int_0^1\Big(t-\frac{1}{2}\Big)\ln\Big(\mathrm{f}^*\Big((1-t)\,\frac{2u+v}{3}+t\frac{u+2v}{3}\Big)\Big)dt\right\}\right|$$
$$\times\left|exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\right.\right.$$
$$\left.\left.\int_0^1\Big(t^{2-2\beta_o}-\frac{1}{2}\Big)\ln\Big(\mathrm{f}^{**}\Big((1-t)\,\frac{2u+v}{3}+t\frac{u+2v}{3}\Big)\Big)dt\right\}\right|$$

$$\times\left|exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left(t-\frac{5}{8}\right)\ln\left(f^*\left((1-t)\,\frac{u+2v}{3}+tv\right)\right)dt\right\}\right|$$

$$\times\left|exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left(t^{2-2\beta_o}-\frac{5}{8}\right)\ln\left(f^{**}\left((1-t)\,\frac{u+2v}{3}+tv\right)\right)dt\right\}\right|$$

$$\le exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{3}{8}\right|\left|\ln\left(f^*\left((1-t)\,u+t\frac{2u+v}{3}\right)\right)\right|dt\right\} \tag{26}$$

$$\times exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|\left|\ln\left(f^{**}\left((1-t)\,u+t\frac{2u+v}{3}\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{1}{2}\right|\left|\ln\left(f^*\left((1-t)\,\frac{2u+v}{3}+t\frac{u+2v}{3}\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\right.$$

$$\left.\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|\left|\ln\left(f^{**}\left((1-t)\,\frac{2u+v}{3}+t\frac{u+2v}{3}\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{5}{8}\right|\left|\ln\left(f^*\left((1-t)\,\frac{u+2v}{3}+tv\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|\left|\ln\left(f^{**}\left((1-t)\,\frac{u+2v}{3}+tv\right)\right)\right|dt\right\}$$

$$= exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{3}{8}\right|\left|\ln\left(f^*\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|\left|\ln\left(f^{**}\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{1}{2}\right|\left|\ln\left(f^*\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)\,(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|\left|\ln\left(f^{**}\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2\,(v-u)^{1+\beta_o}}{9}\int_0^1\left|t-\frac{5}{8}\right|\left|\ln\left(f^*\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|dt\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|\left|\ln\left(f^{**}\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|dt\right\}.$$

Using the multiplicative $P$−convexity of $f^*$ and $f^{**}$, we obtain

$$\left|\ln\left(f^*\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right).$$

Similarly

$$\left|\ln\left(f^{**}\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right),$$

$$\left|\ln\left(f^*\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right),$$

$$\left|\ln\left(f^{**}\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right),$$

$$\left|\ln\left(f^*\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right),$$

and

$$\left|\ln\left(f^{**}\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|\leq\ln\left(f^*(u)f^*(v)\right).$$

Thus, it follows that

$$\mathfrak{M}_{\mathfrak{f}}(u,v)\leq\exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\int_0^1\left(\left|t-\frac{3}{8}\right|+\left|t-\frac{1}{2}\right|+\left|t-\frac{5}{8}\right|\right)dt\left[f^*(u)f^*(v)\right]\right\}$$
$$\times\exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\int_0^1\left(\left|t^{2-2\beta_o}-\frac{3}{8}\right|+\left|t^{2-2\beta_o}-\frac{1}{2}\right|+\left|t^{2-2\beta_o}-\frac{5}{8}\right|\right)dt\left[f^{**}(u)f^{**}(v)\right]\right\}.$$

The desired result can be obtained by noting the following results

$$\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|dt=\frac{1}{3-2\beta_o}\left[1-2\left(\frac{3}{8}\right)^{3-2\beta_o}\right]-\frac{3}{32},$$
$$\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|dt=\frac{1}{3-2\beta_o}\left[1-\left(\frac{1}{2}\right)^{2-2\beta_o}\right],$$

and

$$\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|dt=\frac{1}{3-2\beta_o}\left[1-2\left(\frac{5}{8}\right)^{3-2\beta_o}\right]+\frac{5}{32}.$$

This ends the proof. □

**Theorem 3.4.** *Let $f : \mathrm{I} \subset \Re^+ \to \Re$ be a multiplicative differentiable mapping on $[u, v]$ with $u < v$ and $(\ln f^*)^q$, $(\ln f^{**})^q$ are multiplicative $p-$convex on $[u, v]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following identity hold*

$$\mathfrak{M}_{\mathfrak{f}}(u, v) \leq (\mathrm{f}^{**}(u)\,\mathrm{f}^{**}(v))^{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left((\mathfrak{D}_g(\mathfrak{r}))^{\frac{1}{r}}+(\mathfrak{D}(\mathfrak{r}))^{\frac{1}{r}}+(\mathfrak{D}(\mathfrak{r}))^{\frac{1}{r}}\right)}$$
$$\cdot (\mathrm{f}^{**}(u)\,\mathrm{f}^{**}(v))^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left((\mathfrak{C}_g(\beta_o,\mathfrak{r}))^{\frac{1}{r}}+(\mathfrak{C}(\beta_o,\mathfrak{r}))^{\frac{1}{r}}+(\mathfrak{C}(\beta_o,\mathfrak{r}))^{\frac{1}{r}}\right)}.$$

*where*

$$\mathfrak{C}_g(\beta_o, \mathfrak{r}) = \int_0^1 \left|t^{2-2\beta_o} - \tfrac{3}{8}\right|^r dt,\ \mathfrak{C}(\beta_o, \mathfrak{r}) = \int_0^1 \left|t^{2-2\beta_o} - \tfrac{1}{2}\right|^r dt,$$
$$\mathfrak{C}(\beta_o, \mathfrak{r}) = \int_0^1 \left|t^{2-2\beta_o} - \tfrac{5}{8}\right|^r dt,\quad \mathfrak{D}_g(\mathfrak{r}) = \int_0^1 \left|t - \tfrac{3}{8}\right|^r dt,$$
$$\mathfrak{D}(\mathfrak{r}) = \int_0^1 \left|t - \tfrac{1}{2}\right|^r dt,\ \mathfrak{D}(\mathfrak{r}) = \int_0^1 \left|t - \tfrac{5}{8}\right|^r dt.$$

*Proof.* *Continuing from inequality* (26) *in the proof of Theorem 3.3 and using Hölder's inequality, we have*

$$\begin{aligned}
&\left|\mathfrak{M}_{\mathfrak{f}}(u, v)\right| \\
&\leq exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\int_0^1\left|\ln\left(\mathrm{f}^*\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\right\} \\
&\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\right. \\
&\times\left.\left(\int_0^1\left|\ln\left(\mathrm{f}^{**}\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\right\} \\
&\times exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\left(\int_0^1\left|\ln\left(\mathrm{f}^*\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\right\} \\
&\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\right. \\
&\times\left.\left(\int_0^1\left|\ln\left(\mathrm{f}^{**}\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\right\} \\
&\times exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\int_0^1\left|\ln\left(\mathrm{f}^*\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\right\} \\
&\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\right.
\end{aligned}$$

$$\times\left(\int_0^1\left|\ln\left(f^{**}\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|^s dt\right)^{\frac{1}{s}}\bigg\}. \tag{27}$$

*By using the P−convexity of* $(\ln f^*)^s$ *and* $(\ln f^{**})^s$, *we acquire*

$$\int_0^1\left|\ln\left(f^*\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|^s dt$$
$$\leq\int_0^1 \ln(f^*(u))^s+\ln(f^*(v))^s\,dt=\ln(f^*(u))^s+\ln(f^*(v))^s. \tag{28}$$

*Similarly,*

$$\int_0^1\left|\ln\left(f^{**}\left(\frac{3-t}{3}u+\frac{t}{3}v\right)\right)\right|dt\leq\ln(f^{**}(u))^s+\ln(f^{**}(v))^s, \tag{29}$$

$$\int_0^1\left|\ln\left(f^{*}\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|dt\leq\ln(f^{*}(u))^s+\ln(f^{*}(v))^s, \tag{30}$$

$$\int_0^1\left|\ln\left(f^{**}\left(\frac{2-t}{3}u+\frac{1+t}{3}v\right)\right)\right|dt\leq\ln(f^{**}(u))^s+\ln(f^{**}(v))^s, \tag{31}$$

$$\int_0^1\left|\ln\left(f^{*}\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|dt\leq\ln(f^{*}(u))^s+\ln(f^{*}(v))^s, \tag{32}$$

*and*

$$\int_0^1\left|\ln\left(f^{**}\left(\frac{1-t}{3}u+\frac{2+t}{3}v\right)\right)\right|dt\leq\ln(f^{**}(u))^s+\ln(f^{**}(v))^s. \tag{33}$$

*If we apply the inequalities from* (28)-(33) *into the inequality* (27), *then we obtain*

$$\left|\mathfrak{M}_{\mathfrak{f}}(u,v)\right|\leq exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^*(u))^s+\ln(f^*(v))^s\right)^{\frac{1}{s}}\right\}$$
$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))^s+\ln(f^{**}(v))^s\right)^{\frac{1}{s}}\right\}$$
$$\times exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^*(u))^s+\ln(f^*(v))^s\right)^{\frac{1}{s}}\right\}$$
$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))^s+\ln(f^{**}(v))^s\right)^{\frac{1}{s}}\right\}$$
$$\times exp\left\{\frac{\beta_o{}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^*(u))^s+\ln(f^*(v))^s\right)^{\frac{1}{s}}\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))^s+\ln(f^{**}(v))^s\right)^{\frac{1}{s}}\right\}.$$

*By the use of* $A^s + B^s \leq (A+B)^s$ *for* $A \geq 0, B \geq 0$ *with* $s \geq 1$*, we have that*

$$\left|\mathfrak{M}_{\mathfrak{f}}(u,v)\right| \leq exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{*}(u))+\ln(f^{*}(v))\right)\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))+\ln(f^{**}(v))\right)\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{*}(u))+\ln(f^{*}(v))\right)\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))+\ln(f^{**}(v))\right)\right\}$$

$$\times exp\left\{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\left(\int_0^1\left|t-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{*}(u))+\ln(f^{*}(v))\right)\right\}$$

$$\times exp\left\{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\left(\ln(f^{**}(u))+\ln(f^{**}(v))\right)\right\}$$

$$\leq e^{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\left(\left(\int_0^1\left|t-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\right)(\ln(f^{**}(u))+\ln(f^{**}(v)))}$$

$$\times e^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\left(\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\right)(\ln(f^{**}(u))+\ln(f^{**}(v)))}$$

$$\leq (f^{**}(u)\,f^{**}(v))^{\frac{{\beta_o}^2(v-u)^{1+\beta_o}}{9}\left(\left(\int_0^1\left|t-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\right)}$$

$$\times (f^{**}(u)\,f^{**}(v))^{\frac{(1-\beta_o)(v-u)^{2-\beta_o}}{2.3^{3-2\beta_o}}\left(\left(\int_0^1\left|t^{2-2\beta_o}-\frac{3}{8}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t^{2-2\beta_o}-\frac{1}{2}\right|^r dt\right)^{\frac{1}{r}}+\left(\int_0^1\left|t^{2-2\beta_o}-\frac{5}{8}\right|^r dt\right)^{\frac{1}{r}}\right)}.$$

*The proof is completed.* □

## 4. Applications

*4.1. Special means*

The following notations will be used for special means of two nonnegative numbers $u, v$ with $v > u$:

1. The arithmetic mean

$$u := u(\mathrm{a},\mathrm{b}) = \frac{u+v}{2},$$

2. The geometric mean

$$G := G(a, b) = \sqrt{ab}, \quad a, b \geq 0$$

3. The $p$-logaritmic mean

$$L_p := L_p(a, b) = \begin{cases} \left(\frac{v^{p+1} - u^{p+1}}{(p+1)(v-u)}\right)^{\frac{1}{p}}, & u \neq v, p \in R \setminus \{-1, 0\} \\ u, & u = v \end{cases} ; \; u, v > 0.$$

**Proposition 4.1.** *If* $\forall\; u, v \in [0, \infty)$ *in a way that* $v > u > 0$*, then the subsequent inequality holds*

$$\begin{aligned} &exp\left\{\frac{25}{72}\Big(A(-u, v)\Big(A(u, v)ln(2) + ln(ln(2))\Big)\Big)\right\} \\ &\leq exp\left\{\Big(A(u, v) - L_1(u, v)\Big)ln(2)\right\} \\ &\leq -exp\left\{\frac{25}{72}\Big(A(-u, v)\Big(A(u, v)ln(2) + ln(ln(2))\Big)\Big)\right\} \end{aligned}$$

*Proof.* *One can use Theorem 3.3 in order to show the result that is obtained for considering* $\beta_o = 1$ *and replacing* $\mathfrak{f}(\varkappa)$ *by* $2^{\varkappa}$ *and* $\mathfrak{f}^*(\varkappa)$ *by* $2^{\varkappa}ln(2)$. □

**Example 4.2.** *The Graph 4.3 describes the viability of Proposition 4.1 for various values of* u *and* v *.*

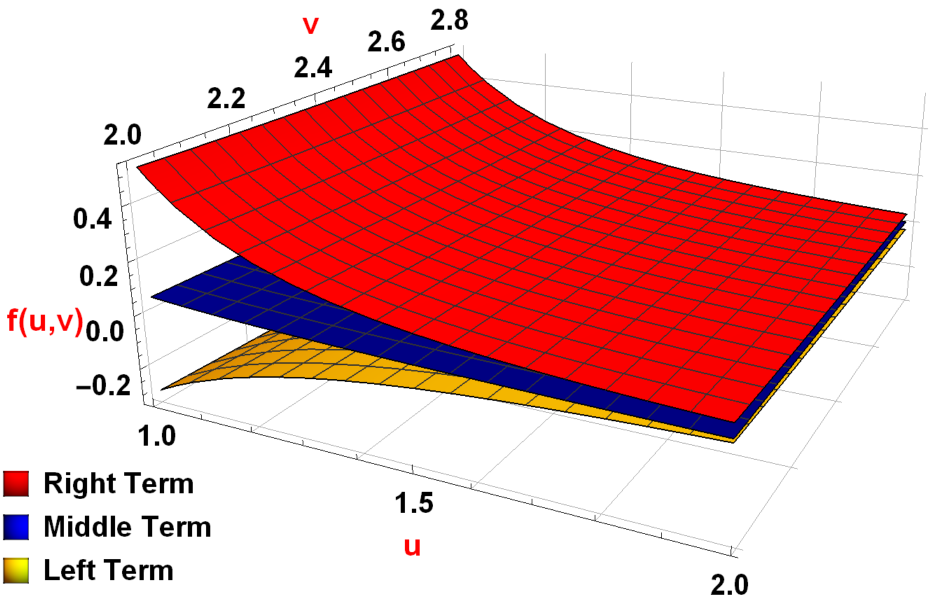



Figure 4.3: Graphical description for $u \in [1, 1.99]$ and $v \in [2, 2.8]$.

*4.2. Modified Bessel function*

Think about the modified Bessel function of type-1 $\mathfrak{J}_{\mathfrak{p}}$ [38].

$$\mathfrak{J}_{\mathfrak{p}}(\varkappa) = \sum_{\mathfrak{n}=0}^{\infty} \frac{(\frac{\varkappa}{2})^{\mathfrak{p}+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\mathfrak{p}+\mathfrak{n}+1)}, \qquad \forall \quad \varkappa \in \mathfrak{R}. \tag{34}$$

Let $\mathfrak{f}_{\mathfrak{p}} : [0, \infty) \to [0, \infty)$ be defined as given in the following for $\mathfrak{p} \geq 2$.

$$\mathfrak{f}_{\mathfrak{p}}(\varkappa) = \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}(\varkappa). \tag{35}$$

$$\mathfrak{f}'_{\mathfrak{p}}(\varkappa) = \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(\varkappa). \tag{36}$$

$$\mathfrak{f}''_{\mathfrak{p}}(\varkappa) = \varkappa^{\mathfrak{p}-1} \mathfrak{J}_{\mathfrak{p}-1}(\varkappa) + \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-2}(\varkappa). \tag{37}$$

$$\mathfrak{f}'''_{\mathfrak{p}}(\varkappa) = 3\varkappa^{\mathfrak{p}-1} \mathfrak{J}_{\mathfrak{p}-2}(\varkappa) + \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-3}(\varkappa). \tag{38}$$

$$\mathfrak{f}^{4}_{\mathfrak{p}}(\varkappa) = 3\varkappa^{\mathfrak{p}-2} \mathfrak{J}_{\mathfrak{p}-2}(\varkappa) + 6\varkappa^{\mathfrak{p}-1} \mathfrak{J}_{\mathfrak{p}-3}(\varkappa) + \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-4}(\varkappa). \tag{39}$$

As $\mathfrak{f}^{4}_{\mathfrak{p}}(\varkappa) > 0, \forall \ \ \mathfrak{p} \geq 1$ and $\varkappa > 0$.
It suggests the convexity of $\mathfrak{f}''_{\mathfrak{p}}(\varkappa)$, $\forall \varkappa \in [0, \infty[$. It is obvious that $exp\{\mathfrak{f}''_{\mathfrak{p}}(\varkappa)\}$ is multiplicatively $P-$convex.

**Proposition 4.3.** *If* $\forall \ u, v \in [0, \infty)$ *in a way that* $v > u > 0$ *and* $\mathfrak{p} \geq 2$, *then the subsequent inequality holds*

$$\begin{aligned} &\left| exp\left\{ \frac{u^{\mathfrak{p}}}{8} \mathfrak{J}_{\mathfrak{p}}(u) + \frac{3}{8}\left(\frac{2u+v}{3}\right)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}\left(\frac{2u+v}{3}\right) + \frac{3}{8}\left(\frac{u+2v}{3}\right)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}\left(\frac{u+2v}{3}\right) \right.\right. \\ &\left.\left. + \frac{v^{\mathfrak{p}}}{8} \mathfrak{J}_{\mathfrak{p}}(v) + \frac{1}{v-u}\left[ v^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}(u) - v^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}(u) \right] \right\} \right| \\ &\leq exp\left\{ \frac{25(v-u)}{288} (u^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(u) + v^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(v)) \right\} \end{aligned} \tag{40}$$

*Proof.* *One can use Theorem 3.3 in order to show the result obtained when considering* $\mathfrak{B}_o = 1$ *and replacing* $\mathfrak{f}(\varkappa)$ *by* $exp\{\mathfrak{f}_{\mathfrak{p}}(\varkappa)\}$ *and* $\mathfrak{f}^*(\varkappa)$ *by* $exp\{\mathfrak{f}'_{\mathfrak{p}}(\varkappa)\}$. □

**Example 4.4.** *The following graph 4.4 describes the viability of Proposition 4.3 for various values of* u, v *and* $\mathfrak{p}$.

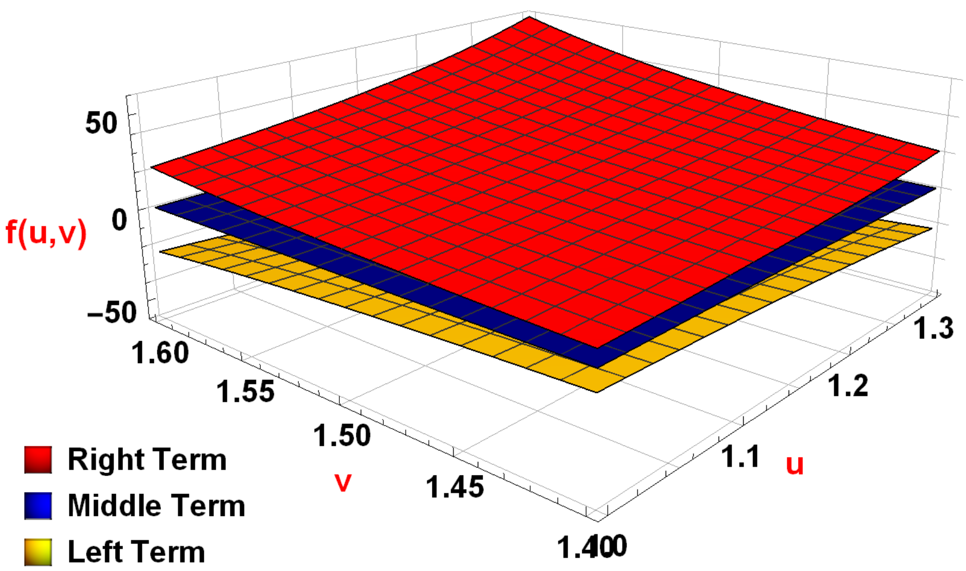



Figure 4.4: Graphical description for $\mathfrak{p} = 2$, $u \in [1, 1.3]$ and $v \in [1.4, 1.6]$.

## 5. Conclusion

This study delves into fractional calculus, a field rich with applications in modeling various natural processes. It highlights the need to continually expand our mathematical tools and frameworks, particularly in the context of fractional integral operators. The paper presents new developments in the theory of $P$-convex functions by introducing a set of $H_rH_d$ type inequalities. These inequalities are generalized using $P_{cap}$ operators, expanding the scope of traditional calculus. The introduction of Lemma 3.1 is significant for deriving new bounds and error estimates related to Newton's type inequalities, providing a deeper understanding of $P$-convex functions.The implications of this research extend to various fields that utilize fractional calculus and convex analysis. The new inequalities and theoretical tools developed can be applied in diverse scientific and engineering contexts, providing valuable information on the properties and behaviors of $P$-convex functions. This study also suggests potential practical applications, particularly in the analysis of special means and type-1 modified Bessel functions. Future research could build on these findings by exploring similar inequalities for other classes of convex functions. There is also a potential to refine the established inequalities, improving the precision and applicability of these mathematical tools. Furthermore, applying these results to practical scenarios across different fields such as physics, biology, and economics could yield further valuable insights and advances. This research contributes significantly to the theoretical landscape of fractional calculus and convex analysis. By introducing new mathematical frameworks and extending existing theories, the study lays the foundations for future explorations and applications. The findings are expected to inspire further research and provide a robust foundation for ongoing developments in these dynamic areas of mathematics. The study's insights are particularly valuable for both theoretical and applied mathematicians interested in the evolving applications of fractional calculus.

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