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On Lupaş-Kantorovich operators with Riemann-Liouville fractional integral

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Abstract. In this note, we deal with Riemann-Liouville type Fractional Lupaş-Kantorovich operators of order α, which introduces a new sequence of positive linear operators with fractional integration. Rate of convergence using modulus of continuity and approximate the Lipschitz class function using these new sequences of positive linear operators are obtained. We also discussed that error estimation in Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5) is better than Lupaş-Kantorovich operators (3). It's weighted approximation properties and Voronovskaja's type approximation theorems are also discussed. Approximation properties of the bivariate Riemann-Liouville fractional integral type Lupaş-Kantorovich operators are introduced and discussed at the end.

1. Introduction

In the year 1995, Lupaş [23] constructed the following operators for the function $f \in C([0,\infty))$

$$L_n(f;x) = \sum_{k=0}^{\infty} \frac{(1-a)^{nx} a^k (nx)_k}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

where $(nx)_0 = 1$ and for $k \geq 1$, $(nx)_k = nx(nx+1)(nx+2)\cdots(nx+k-1)$. Here $C([0,\infty))$ is the set of all real valued continuous functions defined on $[0,\infty)$.

Agratini [4], takes $a = \frac{1}{2}$ and introduces the following operators as

$$L_n(f;x) = \sum_{k=0}^{\infty} \frac{2^{-nx}(nx)_k}{k!2^k} f\left(\frac{k}{n}\right). \tag{2}$$

In order to obtain approximation process in spaces of integrable functions, Agratini [4, 5] introduced the following Kantorovich type operators as

$$K_n(f;x) = n \sum_{k=0}^{\infty} l_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt,$$

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which can be rewritten as

$$K_n(f;x) = \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 f\left(\frac{k+t}{n}\right) dt, \tag{3}$$

where

$$l_{n,k}(x) = \frac{2^{-nx}(nx)_k}{k!2^k}. \tag{4}$$

Since 1995, the Lupaş operators (2) have been discussed by many researchers [1, 6, 7, 13, 14, 16–18, 26, 33–35, 37]. The generalized Jain operators as variant of the Lupaş operators defined by (2) was introduced by Patel & Mishra [28] and further modification was introduced in [9, 27, 29].

Fractional calculus inspires various researchers due to its vast application in real world problem. One of the most important concepts in fractional calculus, the Riemann-Liouville fractional integral generalizes classical notions of integration. Fractional differential equations refer to the equations that contain a derivative or integral of fractional order. The Riemann-Liouville fractional integral plays a key role in solving these equations. Equations of this type arise in a wide range of applications, including electromagnetism, wave propagation, heat conduction and diffusion processes. The Riemann-Liouville fractional integral describes a generalization of the theory of integrals and derivatives. It helps formulate fractional versions of classical theorems and generalizes the idea of differentiation and integration. In the field of economics and finance, the Riemann-Liouville fractional integral has been applied to model complex financial systems and price motion with long-range dependencies or non-local dynamics. The Riemann Liouville fractional integrals were discussed in [11, 20, 21, 32, 36] and studies some inequality using Riemann-Liouville fractional integrals.

Rashid *et al.* [30] introduced an approximate analytical view of physical and biological models in the setting of Caputo operator via the Elzaki transform decomposition method. In 2024, Juraev *et al.* [19] studied the approximate solution of the Cauchy problem for the Helmholtz equation on the plane. Agarwal *et al.* discussed recent trends in fractional calculus and it's applications [3] and fractional differential equations [2]. Later, application of general Lagrange scaling functions for evaluating the approximate solution time-fractional diffusion-wave equations is discussed in [31].

Depending on the context and requirements of the problem under consideration, various definitions of fractional integrals, such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Fourier-based, and Convolution-based, allow for diverse viewpoints and applications. The Riemann-Liouville fractional integral uses the Gamma function to directly expand the classical integral. The Caputo fractional integral is similar to the Riemann-Liouville fractional integral, but it is frequently used in fractional differential equations because it is more convenient with beginning conditions. The Grünwald-Letnikov fractional integral is a discrete approximation used in numerical methods and simulations. The Fourier-based fractional integral is defined by the Fourier transform; this form is very useful in signal processing and spectral approaches. Convolution-based fractional integrals are defined using convolution, and they are important for memory systems and solving fractional diffusion problems. Other fractional integrals like Caputo, Grünwald-Letnikov, Fourier-based, and Convolution-based have not been explored to study approximation properties of positive linear operators like Bernstein-Kantorovich, Baskakov-Kantorovich, Szasz-Mirakjan-Kantorovich, and Lupaş-Kantorovich.

In [24], Mahhmudov and Kara introduced the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators and discussed the order of convergence, weighted approximation properties, and Voronovskaja type results. They also constructed the bivariate Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators and discussed their approximation properties. The Riemann Liouville type fractional Bernstein-Kantorovich operators was introduced in [10] and they discussed approximation properties and rate of convergence. Erdem *et al.* [10], discussed affine functions from the Riemann-Liouville type fractional Bernstein-Kantorovich operators and the bivariate case of Riemann-Liouville type fractional Bernstein-Kantorovich operators. Also, Parmar and Patel [25] discussed On Riemann-Liouville Type Modified Fractional Baskakov-Kantorovich Operators.

We need the following definition of the Riemann-Liouville fractional integral of order $\alpha > 0$ for further discussion

$$_{a^+}I_x^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad (Re(\alpha) > 0, x > a).$$

Motivated by [10, 24, 25], we introduce the Riemann-Liouville type fractional Lupaş-Kantorovich (RLLK) operators of order $\alpha > 0$ as

$$L_n^{\alpha}(f;x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} f\left(\frac{k+t}{n}\right) dt, \tag{5}$$

where $l_{n,k}(x)$ defined in (4).
We note that, in particular if $\alpha = 1$, then operators (5) become Lupaş-Kantorovich operators (3).
In this note, we established approximation properties of the Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5). First, we obtain the moments and central moment of the operators (5). In Section 2, we obtain the rate of convergence using the modulus of continuity and approximate the Lipschitz class function using a sequence of positive linear operators (5). We also argued that error estimation in Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5) is superior to the Lupaş-Kantorovich operators (3). i.e., the rate of convergence improves. It's weighted approximation properties are obtained in the next section. In Section 4, we established Voronovskaja's type approximation theorems. In Section 5, we introduced and discussed the approximation properties of the bivariate Riemann-Liouville fractional integral type Lupaş-Kantorovich operators.

It is clear that the Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5) is positive and linear. To get the approximation properties of (5), we need the following lemmas:

Lemma 1.1. *([4]) Let $e_j(t) = t^j$, $j = 1, 2, 3, 4$, then for given $x \in [0, \infty)$, moments of the Lupaş operators* (2) *are given by*

1. $L_n(e_0; x) = 1;$
2. $L_n(e_1; x) = x;$
3. $L_n(e_2; x) = x^2 + \frac{2x}{n};$
4. $L_n(e_3; x) = x^3 + \frac{6x^2}{n} + \frac{6x}{n^2};$
5. $L_n(e_4; x) = x^4 + \frac{12x^3}{n} + \frac{36x^2}{n^2} + \frac{26x}{n^3}.$

Lemma 1.2. *Let $\alpha \in (0, \infty)$, $n \in \mathbb{N}$, $e_j(t) = t^j$, $j = 0, 1, 2$, then for given $x \in [0, \infty)$, we have*

1. $L_n^{\alpha}(e_0; x) = 1;$
2. $L_n^{\alpha}(e_1; x) = x + \frac{1}{n(\alpha+1)};$
3. $L_n^{\alpha}(e_2; x) = x^2 + \frac{2(\alpha+2)x}{n(\alpha+1)} + \frac{2}{n^2(\alpha+1)(\alpha+2)}.$

Proof. Using beta-gamma relation $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, we get

$$L_n^{\alpha}(e_0; x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} t^{1-1} dt = \alpha \sum_{k=0}^{\infty} l_{n,k}(x) B(\alpha, 1) = 1.$$

For e_1, we obtain

$$
\begin{aligned}
L_n^{\alpha}(e_1;x) &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left(\frac{k+t}{n}\right) dt \\
&= \frac{\alpha}{n} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} k dt + \frac{\alpha}{n} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} t dt \\
&= \alpha \sum_{k=0}^{\infty} l_{n,k}(x) \frac{k}{n} B(\alpha,1) + \frac{\alpha}{n} \sum_{k=0}^{\infty} l_{n,k}(x) B(\alpha,2) \\
&= x + \frac{1}{n(\alpha+1)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
L_n^{\alpha}(e_2;x) &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left(\frac{k+t}{n}\right)^2 dt \\
&= \frac{\alpha}{n^2} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} k^2 dt + \frac{2\alpha}{n^2} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} k t dt + \frac{\alpha}{n^2} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} t^2 dt \\
&= \alpha \sum_{k=0}^{\infty} l_{n,k}(x) \frac{k^2}{n^2} B(\alpha,1) + \frac{2\alpha}{n} \sum_{k=0}^{\infty} l_{n,k}(x) \frac{k}{n} B(\alpha,2) + \frac{\alpha}{n^2} \sum_{k=0}^{\infty} l_{n,k}(x) B(\alpha,3) \\
&= \left(x^2 + \frac{2x}{n}\right) + \frac{2x}{(\alpha+1)n} + \frac{2}{n^2(\alpha+1)(\alpha+2)} \\
&= x^2 + \frac{2(\alpha+2)x}{n(\alpha+1)} + \frac{2}{n^2(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

This concludes the proof of the Lemma 1.2. □

The following Lemma gives relations between the Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5) and the Lupaş operators (2).

Lemma 1.3. *Let $\alpha \in (0,\infty)$, $n \in \mathbb{N}$, then, for given $x \in [0,\infty)$, we have the following:*

$$
L_n^{\alpha}(e_j;x) = \frac{\Gamma(\alpha+1)}{n^j} \sum_{c=0}^{j} \binom{j}{c} \frac{\Gamma(j-c+1)}{\Gamma(\alpha+j-c+1)} n^c L_n(e_c;x),
$$

where $e_j(t) = t^j$, $\forall j \in \mathbb{N} \cup \{0\}$ and L_n is the Lupaş operators, defined in (2).

Proof.

$$
L_n^{\alpha}(e_j;x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left(\frac{k+t}{n}\right)^j dt
$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)n^j}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^1 (1-t)^{\alpha-1}\sum_{c=0}^{j}\binom{j}{c}k^c t^{j-c}dt$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)n^j}\sum_{k=0}^{\infty} l_{n,k}(x)\sum_{c=0}^{j}\binom{j}{c}k^c\int_0^1 (1-t)^{\alpha-1}t^{j-c+1-1}dt$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)n^j}\sum_{k=0}^{\infty} l_{n,k}(x)\sum_{c=0}^{j}\binom{j}{c}k^c B(\alpha, j-c+1)$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)n^j}\sum_{c=0}^{j}\binom{j}{c}\frac{\Gamma(\alpha)\Gamma(j-c+1)}{\Gamma(\alpha+j-c+1)}n^c\sum_{k=0}^{\infty} l_{n,k}(x)\left(\frac{k}{n}\right)^c$$

$$= \frac{\Gamma(\alpha+1)}{n^j}\sum_{c=0}^{j}\binom{j}{c}\frac{\Gamma(j-c+1)}{\Gamma(\alpha+j-c+1)}n^c L_n(e_c;x),$$

which completes the proof of the Lemma 1.3. □

Corollary 1.4. *Let $e_j(t) = t^j$, $j = 3, 4$, then for given $x \in [0,\infty)$, we get*

1. $L_n^\alpha(e_3;x) = x^3 + \frac{(6\alpha+9)x^2}{n(\alpha+1)} + \frac{(6\alpha^2+24\alpha+30)x}{n^2(\alpha+1)(\alpha+2)} + \frac{6}{n^3(\alpha+1)(\alpha+2)(\alpha+3)}$;
2. $L_n^\alpha(e_4;x) = x^4 + \frac{(12\alpha+16)x^3}{n(\alpha+1)} + \frac{(36\alpha^2+132\alpha+132)x^2}{n^2(\alpha+1)(\alpha+2)} + \frac{(26\alpha^3+180\alpha^2+430\alpha+396)x}{n^3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{24}{n^4(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$.

Proof. Using Lemma 1.1 and 1.3, one can obtain. □

Lemma 1.5. *Let $\alpha \in (0,\infty)$, $n \in \mathbb{N}$, then for given $x \in [0,\infty)$, we have*

$$|L_n^\alpha(f;x)| \le \|f\|.$$

Proof. Using integral inequality, we get

$$|L_n^\alpha(f;x)| \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^1 (1-t)^{\alpha-1}\left|f\left(\frac{k+t}{n+1}\right)\right|dt$$

$$\le \|f\|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^1 (1-t)^{\alpha-1}dt$$

$$= \|f\|.$$

Thus, this completes the proof of lemma 1.5. □

In the following Lemma, we have calculated first four central moment of the operators (5).

Lemma 1.6. *Let $\alpha \in (0,\infty)$, $n \in \mathbb{N}$, $(e_1 - x)^j(t) = (t-x)^j$, $j = 1,2,3,4$, then for given $x \in [0,\infty)$, we get*

1. $L_n^\alpha(e_1 - x;x) = \frac{1}{n(\alpha+1)} = \zeta_{n,\alpha}(x)$;
2. $L_n^\alpha((e_1 - x)^2;x) = \frac{2x}{n} + \frac{2}{(\alpha+1)(\alpha+2)n^2} = \beta_{n,\alpha}(x)$;
3. $L_n^\alpha((e_1 - x)^3;x) = \frac{(6\alpha^2+24\alpha+24)x}{n^2(\alpha+1)(\alpha+2)} + \frac{6}{n^3(\alpha+1)(\alpha+2)(\alpha+3)}$;
4. $L_n^\alpha((e_1 - x)^4;x) = \frac{(12\alpha^2+36\alpha+24)x^2}{n^2(\alpha+1)(\alpha+2)} + \frac{(26\alpha^3+180\alpha^2+430\alpha+372)x}{n^3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{24}{n^4(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$.

Proof. Using lemma 1.2, we get

$$L_n^\alpha(e_1 - x;x) = L_n^\alpha(e_1;x) - xL_n^\alpha(e_0;x) = x + \frac{1}{n(\alpha+1)} - x(1) = \frac{1}{n(\alpha+1)}.$$

For $(e_1 - x)^2$, we obtain

$$\begin{aligned} L_n^\alpha((e_1-x)^2;x) &= L_n^\alpha(t^2;x) - 2xL_n^\alpha(t;x) + x^2L_n^\alpha(1;x) \\ &= x^2 + \frac{2(\alpha+2)x}{(\alpha+1)n} + \frac{2}{n^2(\alpha+1)(\alpha+2)} - 2x\left(x + \frac{1}{(\alpha+1)n}\right) + x^2 \\ &= \frac{2x}{n} + \frac{2}{(\alpha+1)(\alpha+2)n^2}. \end{aligned}$$

Using lemma 1.2 and corollary 1.4, we achieve

$$\begin{aligned} L_n^\alpha((e_1-x)^3;x) &= L_n^\alpha(t^3;x) - 3xL_n^\alpha(t^2;x) + 3x^2L_n^\alpha(t;x) - x^3L_n^\alpha(1;x) \\ &= \frac{(6\alpha^2+24\alpha+24)x}{n^2(\alpha+1)(\alpha+2)} + \frac{6}{n^3(\alpha+1)(\alpha+2)(\alpha+3)}. \end{aligned}$$

Similarly for $(e_1 - x)^4$, we have

$$\begin{aligned} L_n^\alpha((e_1-x)^4;x) &= L_n^\alpha(t^4;x) - 4xL_n^\alpha(t^3;x) + 6x^2L_n^\alpha(t^2;x) - 4x^3L_n^\alpha(t;x) + x^4L_n^\alpha(1;x) \\ &= \frac{(12\alpha^2+36\alpha+24)x^2}{n^2(\alpha+1)(\alpha+2)} + \frac{(26\alpha^3+180\alpha^2+430\alpha+372)x}{n^3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{24}{n^4(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}. \end{aligned}$$

This completes the proof of Lemma 1.6. □

Corollary 1.7. *Let $f \in C([0,\infty))$ and $B > 0$, then sequence of operators $L_n^\alpha(f)$ converges to f uniformly on $[0, B]$.*

Proof. Using Lemma 1.2, we get

$$\lim_{n\to\infty} L_n^\alpha(e_0;x) = 1.$$

For e_1, we have

$$\lim_{n\to\infty} L_n^\alpha(e_1;x) = \lim_{n\to\infty}\left(x + \frac{1}{n(\alpha+1)}\right) = x.$$

Similarly for e_2, we obtain

$$\lim_{n\to\infty} L_n^\alpha(e_2;x) = \lim_{n\to\infty}\left(x^2 + \frac{2(\alpha+2)x}{n(\alpha+1)} + \frac{2}{n^2(\alpha+1)(\alpha+2)}\right) = x^2.$$

Hence, by the Korovkin theorem the sequence of operators $L_n^\alpha(f)$ converges to f uniformly on $[0, B]$. □

Lemma 1.8. *Let $\alpha \in (0,\infty)$, $n \in \mathbb{N}$, $(e_1 - x)^j(t) = (t - x)^j$, $j = 1, 2, 3, 4$, then for given $x \in [0,\infty)$, we have*

1. $\lim_{n\to\infty} nL_n^\alpha(e_1 - x;x) = \frac{1}{\alpha+1}$;
2. $\lim_{n\to\infty} nL_n^\alpha((e_1 - x)^2;x) = 2x$;
3. $\lim_{n\to\infty} nL_n^\alpha((e_1 - x)^3;x) = 0$;
4. $\lim_{n\to\infty} nL_n^\alpha((e_1 - x)^4;x) = 0$.

Proof. Using Lemma 1.6, we get

$$\lim_{n\to\infty} nL_n^\alpha(e_1 - x;x) = \lim_{n\to\infty} n\left(\frac{1}{(\alpha+1)n}\right) = \frac{1}{\alpha+1}.$$

For $(e_1 - x)^2$, we obtain

$$\lim_{n\to\infty} nL_n^{\alpha}((e_1 - x)^2; x) = \lim_{n\to\infty} n\left(\frac{2x}{n} + \frac{2}{(\alpha+1)(\alpha+2)n^2}\right) = 2x.$$

Similarly,

$$\lim_{n\to\infty} nL_n^{\alpha}((e_1 - x)^3; x) = \lim_{n\to\infty} nL_n^{\alpha}((e_1 - x)^4; x) = 0.$$

Thus, the proof of the Lemma 1.8 is concluded. □

2. Approximation properties of the Riemann-Liouville fractional integral type Lupaş- Kantorovich operators

To get approximation properties of the Riemann-Liouville fractional integral type Lupaş- Kantorovich operators, we need the following definitions.

Let $C_B([0,\infty))$ be the space of all real valued continuous and bounded functions f on $[0,\infty)$, then for $f \in C_B[0,\infty)$, the modulus of continuity is defined as

$$w(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Let $C_B^2([0,\infty)) = \{g \in C_B([0,\infty)) : g', g'' \in C_B([0,\infty))\}$, then Peetre's K-functional $K_2(f;\delta)$ is defined as

$$K_2(f;\delta) = \inf_{g\in C_B^2([0,\infty))}\{\|f-g\| + \delta\|g''\|\},\ \delta > 0.$$

The second order modulus of smoothness for $f \in C_B([0,\infty))$ is given by

$$w_2(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Lemma 2.1. *Let $\alpha \in (0,\infty)$, $f \in C_B([0,\infty))$ and $g \in C_B^2([0,\infty))$, then for given $x \in [0,\infty)$, we get*

$$|\hat{L_n^{\alpha}}(g;x) - g(x)| \leq (\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)\|g''\|,$$

where $\zeta_{n,\alpha}(x)$, $\beta_{n,\alpha}(x)$ are defined in Lemma 1.6 and $\hat{L_n^{\alpha}}(f;x) = L_n^{\alpha}(f;x) + f(x) - f(L_n^{\alpha}(t;x))$.

Proof. Let $g \in C_B^2([0,\infty))$, then by Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\hat{L_n^{\alpha}}$ on both sides, we get

$$\hat{L_n^{\alpha}}(g;x) = \hat{L_n^{\alpha}}(g(x);x) + \hat{L_n^{\alpha}}((t-x)g'(x);x) + \hat{L_n^{\alpha}}\left(\int_x^t (t-v)g''(v)dv;x\right). \tag{6}$$

Clearly $\hat{L_n^{\alpha}}$ is linear operators, so

$$\hat{L_n^{\alpha}}(g(x);x) = g(x)\hat{L_n^{\alpha}}(1;x) = g(x).$$

Again, by using linearity of $\hat{L_n^{\alpha}}$, we get

$$\hat{L_n^{\alpha}}((t-x)g'(x);x) = g'(x)[\hat{L_n^{\alpha}}(t;x) - \hat{L_n^{\alpha}}(x;x)] = g'(x)[L_n^{\alpha}(t;x) + x - L_n^{\alpha}(t;x) - x(1)] = 0.$$

Furthermore,

$$\hat{L_n^{\alpha}}\left(\int_x^t (t-v)g''(v)dv;x\right) = L_n^{\alpha}\left(\int_x^t (t-v)g''(v)dv;x\right) - \int_x^{L_n^{\alpha}(t;x)} (L_n^{\alpha}(t;x)-v)g''(v)dv.$$

So (6) becomes

$$\hat{L_n^{\alpha}}(g;x) = g(x) + L_n^{\alpha}\left(\int_x^t (t-v)g''(v)dv;x\right) - \int_x^{L_n^{\alpha}(t;x)} (L_n^{\alpha}(t;x)-v)g''(v)dv.$$

Using triangle and integral inequality, we get

$$|\hat{L_n^{\alpha}}(g;x) - g(x)| \leq L_n^{\alpha}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right) + \int_x^{L_n^{\alpha}(t;x)} |(L_n^{\alpha}(t;x)-v)g''(v)|dv. \tag{7}$$

Now,

$$\int_x^t |(t-v)g''(v)|dv \leq \|g''\| \int_x^t (t-v)dv \leq (t-x)^2\|g''\|.$$

Hence,

$$L_n^{\alpha}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right) \leq L_n^{\alpha}((t-x)^2;x)\|g''\|,$$

and

$$\int_x^{L_n^{\alpha}(t;x)} |(L_n^{\alpha}(t;x)-v)g''(v)|dv \leq (L_n^{\alpha}(t;x)-x)^2\|g''\|.$$

Finally, using (7), we get

$$|\hat{L_n^{\alpha}}(g;x) - g(x)| \leq (L_n^{\alpha}((t-x)^2;x) + (L_n^{\alpha}(t-x;x))^2)\|g''\| = (\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)\|g''\|.$$

Thus, the proof is complete. □

Theorem 2.2. *Let $\alpha \in (0,\infty)$, $n \in \mathbb{N}$, then for any $f \in C_B([0,\infty))$ and $x \geq 0$ there exists a real number $E > 0$ such that*

$$|L_n^{\alpha}(f;x) - f(x)| \leq E w_2\left(f; \frac{\sqrt{\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2}}{2}\right) + w(f;\zeta_{n,\alpha}(x)),$$

where $\zeta_{n,\alpha}(x)$, $\beta_{n,\alpha}(x)$ are defined in the lemma 1.6.

Proof. Using triangle inequality, we get

$$|\hat{L_n^{\alpha}}(f;x)| \leq |L_n^{\alpha}(f;x)| + |f(x)| + |f(L_n^{\alpha}(t;x))| = 3\|f\|.$$

Hence

$$|\hat{L_n^{\alpha}}(f-g;x)| \leq 3\|f-g\|.$$

By Lemma 2.1, we have

$$|\hat{L_n^{\alpha}}(g;x) - g(x)| \leq (\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)\|g''\|.$$

Using the property of modulus of continuity, we obtain

$$|f(L_n^{\alpha}(t;x)) - f(x)| \leq w(f; L_n^{\alpha}(t-x;x)).$$

Now

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &= |\hat{L_n^\alpha}(f;x) - f(x) + f(L_n^\alpha(t;x)) - f(x)| \\ &\leq |\hat{L_n^\alpha}(f;x) - \hat{L_n^\alpha}(g;x)| + |\hat{L_n^\alpha}(g;x) - g(x)| + |f(x) - g(x)| + |f(L_n^\alpha(t;x)) - f(x)| \\ &\leq 4\|f - g\| + (\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)\|g''\| + w(f;\zeta_{n,\alpha}(x)). \end{aligned}$$

Taking the infimum over all $g \in C_B^2([0,\infty))$, we get

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &\leq 4 \inf_{g\in C_B^2[0,\infty)} \left[\|f-g\| + \frac{(\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)}{4} \|g''\| \right] + w(f;\zeta_{n,\alpha}(x)) \\ &= 4K_2\left(f; \frac{(\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2)}{4}\right) + w(f;\zeta_{n,\alpha}(x)). \end{aligned}$$

Using [12, theorem 2.4], there exists a constant $E > 0$ such that $K_2(f;\delta) \leq Ew_2(f;\sqrt{\delta})$, we have

$$|L_n^\alpha(f;x) - f(x)| \leq Ew_2\left(f; \frac{\sqrt{\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2}}{2}\right) + w(f;\zeta_{n,\alpha}(x)).$$

Thus, we obtain the desired result. □

Remark 1: For $\alpha = 1$, Theorem 2.2 becomes

$$|K_n(f;x) - f(x)| \leq Ew_2\left(f; \frac{1}{2}\sqrt{\frac{2x}{n} + \frac{1}{3n^2} + (\frac{1}{2n})^2}\right) + w\left(f; \frac{1}{2n}\right), \tag{8}$$

where K_n is Lupaş-Kantorovich operators [4, 5].

Remark 2: We claim that Theorem 2.2 outperforms (8) in terms of error estimation for the larger α. To get a more accurate approximation, show that $\frac{1}{2}\sqrt{\frac{2x}{n} + \frac{1}{3n^2} + (\frac{1}{2n})^2} \geq \frac{\sqrt{\beta_{n,\alpha}(x)+(\zeta_{n,\alpha}(x))^2}}{2}$. We accomplish

$$\begin{aligned} & \frac{1}{4}\left(\frac{2x}{n} + \frac{1}{3n^2} + (\frac{1}{2n})^2\right) - \frac{\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2}{4} \\ = & \frac{1}{4}\left(\frac{2x}{n} + \frac{1}{3n^2} + \left(\frac{1}{2n}\right)^2 - \left(\frac{2x}{n} + \frac{2}{(\alpha+1)(\alpha+2)n^2} + \left(\frac{1}{(\alpha+1)n}\right)^2\right)\right) \\ = & \frac{1}{4n^2}\left(\frac{\alpha^2 + 3\alpha - 4}{3(\alpha+1)(\alpha+2)} + \frac{\alpha^2 + 2\alpha - 3}{4(\alpha+1)^2}\right) \\ = & \frac{7\alpha^3 + 28\alpha^2 - \alpha - 34}{48n^2(\alpha+1)^2(\alpha+2)}. \end{aligned}$$

Hence, for a larger value of α, i.e., for $\alpha \geq 1$, we get

$$\frac{1}{4}\left(\frac{2x}{n} + \frac{1}{3n^2} + (\frac{1}{2n})^2\right) - \frac{\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2}{4} \geq 0.$$

Thus, we have

$$\frac{1}{2}\sqrt{\frac{2x}{n} + \frac{1}{3n^2} + (\frac{1}{2n})^2} \geq \frac{\sqrt{\beta_{n,\alpha}(x) + (\zeta_{n,\alpha}(x))^2}}{2}.$$

Lipschitz class for all $x_1, x_2 \in [0,\infty)$ is defined as

$$Lip_T(\lambda) = \{f \in C_B[0,\infty) : |f(x_1) - f(x_2)| \leq T|x_1 - x_2|^\lambda\},$$

where $0 < \lambda \leq 1$ and $T > 0$ is constant.

Theorem 2.3. *For every $f \in Lip_T(\lambda)$, we have*

$$|L_n^\alpha(f;x) - f(x)| \le T((\beta_{n,\alpha}(x))^{\frac{\lambda}{2}} + 2(d(x,X))^\lambda),$$

where T is constant, $d(x,X) = \inf\{|t-x| : t \in X\}$ and $\beta_{n,\alpha}(x)$ is defined in Lemma 1.6.

Proof. Let $\overline{X}$ be the closure of X, then by property of infimum there is atleast one point $t_1 \in \overline{X}$ such that $d(x,X) = |x - t_1|$.
Using definition of Lipschitz class, we obtain

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &= |L_n^\alpha(f(t);x) - L_n^\alpha(f(t_1);x) + L_n^\alpha(f(t_1);x) - f(x)| \\ &\le |L_n^\alpha(f(t) - f(t_1);x)| + |f(t_1) - f(x)| \\ &\le T[L_n^\alpha(|t-t_1|^\lambda;x) + |x-t_1|^\lambda] \\ &\le T[L_n^\alpha(|t-x|^\lambda + |x-t_1|^\lambda;x) + |x-t_1|^\lambda] \\ &= T[L_n^\alpha(|t-x|^\lambda;x) + L_n^\alpha(|x-t_1|^\lambda;x) + |x-t_1|^\lambda] \\ &= T[L_n^\alpha(|t-x|^\lambda;x) + 2|x-t_1|^\lambda]. \end{aligned}$$

Applying the Hölder's inequality by choosing $p = \frac{2}{\lambda}$ and $q = \frac{2}{2-\lambda}$, we get

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &\le T[(L_n^\alpha((t-x)^2;x))^{\frac{1}{p}}(L_n^\alpha(1;x))^{\frac{1}{q}} + 2|x-t_1|^\lambda] \\ &= T((\beta_{n,\alpha}(x))^{\frac{\lambda}{2}} + 2(d(x,X))^\lambda). \end{aligned}$$

Thus, we achieve the desired result. □

Theorem 2.4. *For every $f \in Lip_T(\lambda)$, we have*

$$|L_n^\alpha(f;x) - f(x)| \le T(\beta_{n,\alpha}(x))^{\frac{\lambda}{2}},$$

where T is constant and $\beta_{n,\alpha}(x)$ is defined in Lemma 1.6.

Proof. Using definition of Lipschitz class, we get

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &\le L_n^\alpha(|f(t) - f(x)|;x) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left| f\left(\frac{k+t}{n+1}\right) - f(x) \right| dt \\ &\le T\alpha \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left| \frac{k+t}{n+1} - x \right|^\lambda dt. \end{aligned}$$

Applying the Hölder's inequality by choosing $p = \frac{2}{\lambda}$ and $q = \frac{2}{2-\lambda}$, we get

$$\begin{aligned} |L_n^\alpha(f;x) - f(x)| &\le T\left(\alpha \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} \left(\frac{k+t}{n+1} - x\right)^2 dt\right)^{\frac{1}{p}} \left(\alpha \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} dt\right)^{\frac{1}{q}} \\ &= T[L_n^\alpha((t-x)^2;x)]^{\frac{\lambda}{2}} = T(\beta_{n,\alpha}(x))^{\frac{\lambda}{2}}. \end{aligned}$$

This concludes the proof of Theorem 2.4. □

B. Lenze [22] introduced Lipschitz-type maximal function of order λ as

$$\hat{w_\lambda}(f;x) = \sup_{x,t\in[0,\infty), t\neq x} \frac{|f(t) - f(x)|}{|t-x|^\lambda}, \ \lambda \in (0,1].$$

Theorem 2.5. *Let $\alpha \in (0,\infty)$, $f \in C_B[0,\infty)$ and $\lambda \in (0,1]$, then for all $x \geq 0$, we have*

$$|L_n^\alpha(f;x) - f(x)| \leq \hat{w}_\lambda(f;x)(\beta_{n,\alpha}(x))^{\frac{\lambda}{2}},$$

where $\beta_{n,\alpha}(x)$ is defined in Lemma 1.6.

Proof. Using definition of Lipschitz type maximal function, we get

$$|L_n^\alpha(f;x) - f(x)| \leq L_n^\alpha(|f(t) - f(x)|;x) \leq L_n^\alpha(\hat{w}_\lambda(f;x)|t-x|^\lambda;x) = \hat{w}_\lambda(f;x)L_n^\alpha(|t-x|^\lambda;x).$$

Applying Hölder's inequality by choosing $p = \frac{2}{\lambda}$ and $q = \frac{2}{2-\lambda}$, we get

$$|L_n^\alpha(f;x) - f(x)| \leq \hat{w}_\lambda(f;x)[(L_n^\alpha((t-x)^2;x))^{\frac{1}{p}}(L_n^\alpha(1;x))^{\frac{1}{q}}] = \hat{w}_\lambda(f;x)(\beta_{n,\alpha}(x))^{\frac{\lambda}{2}},$$

we obtain the required result. □

3. Weighted approximation

To prove the weighted approximation properties of the operators (5), we need the following set of functions:

Let $B_m([0,\infty))$ be the set of all functions satisfying condition $\frac{|f(x)|}{1+x^m} \leq M_f$, for any given $x \in [0,\infty)$, where M_f is constant depending on f,

$$C_m([0,\infty)) = \{f \in B_m([0,\infty)) \cap C([0,\infty))\},$$

and

$$C_m^*([0,\infty)) = \{f \in C_m([0,\infty)) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty\}.$$

The norm $||f||_m$ on $C_m^*([0,\infty))$ is defined as

$$||f||_m = \sup_{x\geq 0} \frac{|f(x)|}{1+x^m} < \infty.$$

Theorem 3.1. *Let $\alpha \in (0,\infty)$, $f \in C_2^*([0,\infty))$, then we obtain*

$$\lim_{n\to\infty} ||L_n^\alpha(f;\cdot) - f(\cdot)||_2 = 0.$$

Proof. Using Lemma 1.2, we get

$$\lim_{n\to\infty} ||L_n^\alpha(e_0;\cdot) - e_0(\cdot)||_2 = 0.$$

Similarly for e_1, we have

$$||L_n^\alpha(e_1;\cdot) - e_1(\cdot)||_2 = \sup_{x\geq 0} \frac{|L_n^\alpha(e_1;x) - x|}{1+x^2} = \sup_{x\geq 0}\left(\frac{1}{1+x^2}\right)\frac{1}{n(\alpha+1)} \leq \frac{1}{n(\alpha+1)}.$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} ||L_n^\alpha(e_1;\cdot) - e_1(\cdot)||_2 = 0.$$

Furthermore,

$$\begin{aligned}||L_n^\alpha(e_2;\cdot)-e_2(\cdot)||_2 &= \sup_{x\geq 0}\frac{|L_n^\alpha(e_2;x)-x^2|}{1+x^2}\\ &= \sup_{x\geq 0}\left(\frac{1}{1+x^2}\right)\left|\frac{2(\alpha+2)x}{n(\alpha+1)}+\frac{2}{n^2(\alpha+1)(\alpha+2)}\right|\\ &\leq \frac{2(\alpha+2)}{n(\alpha+1)}+\frac{2}{n^2(\alpha+1)(\alpha+2)}.\end{aligned}$$

Taking limit on both sides, we obtain

$$\lim_{n\to\infty}||L_n^\alpha(e_2;\cdot)-e_2(\cdot)||_2=0.$$

So for $j=0,1,2$, we have

$$\lim_{n\to\infty}||L_n^\alpha(e_j;\cdot)-e_j(\cdot)||_2=0.$$

Hence, by the Korovkin theorem [15] on weighted approximation, we get

$$\lim_{n\to\infty}||L_n^\alpha(f;\cdot)-f(\cdot)||_2=0.$$

This completes the proof of Theorem 3.1 □

4. Voronovskaja type results

Theorem 4.1. *Let $\alpha\in(0,\infty)$, then for any $f\in C_2^*([0,\infty))$, $f', f''\in C_2^*([0,\infty))$, we have*

$$\lim_{n\to\infty}n[L_n^\alpha(f;x)-f(x)]=\frac{f'(x)}{\alpha+1}+xf''(x).$$

Proof. Using the Taylor's formula, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+r(t,x)(t-x)^2,$$

where $r(\cdot,\cdot)\in C_2^*([0,\infty))$ is a peano form of remainder and $\lim_{t\to x}r(t,x)=0$.
Applying L_n^α operators on both sides, we get

$$L_n^\alpha(f;x)=f(x)+f'(x)L_n^\alpha(t-x;x)+\frac{f''(x)}{2}L_n^\alpha((t-x)^2;x)+L_n^\alpha(r(t,x)(t-x)^2;x).$$

Taking limit on both sides, we obtain

$$\lim_{n\to\infty}n[L_n^\alpha(f;x)-f(x)]=f'(x)\lim_{n\to\infty}nL_n^\alpha(t-x;x)+\frac{f''(x)}{2}\lim_{n\to\infty}nL_n^\alpha((t-x)^2;x)+\lim_{n\to\infty}nL_n^\alpha(r(t,x)(t-x)^2;x).$$

Using Lemma 1.8, we have

$$\lim_{n\to\infty}n[L_n^\alpha(f;x)-f(x)]=f'(x)\left(\frac{1}{\alpha+1}\right)+\frac{f''(x)}{2}(2x)+\lim_{n\to\infty}nL_n^\alpha(r(t,x)(t-x)^2;x). \tag{9}$$

Using Cauchy-Schwarz's inequality, we get

$$L_n^\alpha(r(t,x)(t-x)^2;x)\leq[L_n^\alpha(r^2(t,x);x)]^{\frac{1}{2}}[L_n^\alpha((t-x)^4;x)]^{\frac{1}{2}}.$$

Since $\lim_{n\to\infty}L_n^\alpha(r^2(t,x);x)=r^2(x,x)=0$, So $\lim_{n\to\infty}nL_n^\alpha(r(t,x)(t-x)^2;x)=0$.

Hence, by (9), we obtain

$$\lim_{n\to\infty}n[L_n^\alpha(f;x)-f(x)]=\frac{f'(x)}{\alpha+1}+xf''(x).$$

Thus the proof is completed. □

5. Bivariate Riemann-Liouville fractional integral type Lupaş-Kantorovich operators

Let $\alpha_1, \alpha_2 \in (0, \infty)$ and $n_1, n_2 \in \mathbb{N}$, then $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, the bivariate Riemann-Liouville fractional integral type Lupaş-Kantorovich operators is defined as follows:

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(f; x_1, x_2) = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2)} \sum_{k_1=0}^{\infty} l_{n_1,k_1}(x_1) \sum_{k_2=0}^{\infty} l_{n_2,k_2}(x_2) \times \int_0^1 \int_0^1 (1-t_1)^{\alpha_1-1}(1-t_2)^{\alpha_2-1} f\left(\frac{k_1+t_1}{n_1}, \frac{k_2+t_2}{n_2}\right) dt_1 dt_2, \tag{10}$$

where $l_{n_1,k_1}(x_1) = \frac{2^{-n_1x_1}(n_1x_1)_{k_1}}{k_1! 2^{k_1}}$ and $l_{n_2,k_2}(x_2) = \frac{2^{-n_2x_2}(n_2x_2)_{k_2}}{k_2! 2^{k_2}}$,
$(n_1x_1)_0 = 1$, for $k_1 \geq 1$, $(n_1x_1)_{k_1} = n_1x_1(n_1x_1+1)(n_1x_1+2)...(n_1x_1+k_1-1)$,
$(n_2x_2)_0 = 1$, for $k_2 \geq 1$, $(n_2x_2)_{k_2} = n_2x_2(n_2x_2+1)(n_2x_2+2)...(n_2x_2+k_2-1)$.

If we take $\alpha_1 = \alpha_2 = 1$, then the operators (10) turn into the bivariate Lupaş-Kantorovich operators.

The following Lemma gives relations between the bivariate Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (10) and the Riemann-Liouville fractional integral type Lupaş-Kantorovich operators (5).

Lemma 5.1. *Let $\alpha_1, \alpha_2 \in (0, \infty)$ and $n_1, n_2 \in \mathbb{N}$, then for $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, we get*

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{ji}; x_1, x_2) = L_{n_1}^{\alpha_1}(e_j; x_1) L_{n_2}^{\alpha_2}(e_i; x_2),$$

where $e_{ji}(t_1, t_2) = t_1^j t_2^i$, $e_j(t_1) = t_1^j$, $e_i(t_2) = t_2^i$, $\forall i, j \in \mathbb{N} \cup \{0\}$.

Proof. we note that,

$$\begin{aligned} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{ji}; x_1, x_2) &= \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1)} \sum_{k_1=0}^{\infty} l_{n_1,k_1}(x_1) \int_0^1 (1-t_1)^{\alpha_1-1} \left(\frac{k_1+t_1}{n_1}\right)^j dt_1 \\ &\quad \times \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2)} \sum_{k_2=0}^{\infty} l_{n_2,k_2}(x_2) \int_0^1 (1-t_2)^{\alpha_2-1} \left(\frac{k_2+t_2}{n_2}\right)^i dt_2 \\ &= L_{n_1}^{\alpha_1}(e_j; x_1) L_{n_2}^{\alpha_2}(e_i; x_2), \end{aligned}$$

which completes the proof of the Lemma 5.1. □

Lemma 5.2. *Let $\alpha_1, \alpha_2 \in (0, \infty)$ and $n_1, n_2 \in \mathbb{N}$, then for $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, we obtain*

1. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{00}; x_1, x_2) = 1$;
2. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{10}; x_1, x_2) = x_1 + \frac{1}{n_1(\alpha_1+1)}$;
3. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{01}; x_1, x_2) = x_2 + \frac{1}{n_2(\alpha_2+1)}$;
4. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{20}; x_1, x_2) = x_1^2 + \frac{2(\alpha_1+2)x_1}{(\alpha_1+1)n_1} + \frac{2}{n_1^2(\alpha_1+1)(\alpha_1+2)}$;
5. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{02}; x_1, x_2) = x_2^2 + \frac{2(\alpha_2+2)x_2}{(\alpha_2+1)n_2} + \frac{2}{n_2^2(\alpha_2+1)(\alpha_2+2)}$;

where $e_{ji}(t_1, t_2) = t_1^j t_2^i$, $0 \leq j + i \leq 2$.

Proof. Using Lemma 5.1 and 1.2, we get

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{00};x_1,x_2) = L_{n_1}^{\alpha_1}(e_0;x_1)L_{n_2}^{\alpha_2}(e_0;x_2) = 1.$$

For e_{10}, we have

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{10};x_1,x_2) = L_{n_1}^{\alpha_1}(e_1;x_1)L_{n_2}^{\alpha_2}(e_0;x_2) = x_1 + \frac{1}{n_1(\alpha_1+1)}.$$

Similarly,

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{01};x_1,x_2) = x_2 + \frac{1}{n_2(\alpha_2+1)}.$$

For e_{20}, we obtain

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{20};x_1,x_2) = L_{n_1}^{\alpha_1}(e_2;x_1)L_{n_2}^{\alpha_2}(e_0;x_2) = x_1^2 + \frac{2(\alpha_1+2)x_1}{(\alpha_1+1)n_1} + \frac{2}{n_1^2(\alpha_1+1)(\alpha_1+2)}.$$

Similarly,

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{02};x_1,x_2) = x_2^2 + \frac{2(\alpha_2+2)x_2}{(\alpha_2+1)n_2} + \frac{2}{n_2^2(\alpha_2+1)(\alpha_2+2)}.$$

Hence, the proof of Lemma 5.2 is now complete. □

Lemma 5.3. *Let $\alpha_1,\alpha_2 \in (0,\infty)$ and $n_1,n_2 \in \mathbb{N}$, then for $(x_1,x_2) \in [0,\infty)\times[0,\infty)$, we get*

1. $L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{10}-x_1)^j;x_1,x_2) = L_{n_1}^{\alpha_1}((e_1-x_1)^j;x_1)$;
2. $L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{01}-x_2)^i;x_1,x_2) = L_{n_2}^{\alpha_2}((e_1-x_2)^i;x_2)$;

where $(e_{10}-x_1)^j(t_1,t_2) = (t_1-x_1)^j$, $(e_{01}-x_2)^i(t_1,t_2) = (t_2-x_2)^i$, $\forall i, j \in \mathbb{N}\cup\{0\}$.

Proof. we note that

$$\begin{aligned} L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{10}-x_1)^j;x_1,x_2) &= \alpha_1 \sum_{k_1=0}^{\infty} l_{n_1,k_1}(x_1)\int_0^1 (1-t_1)^{\alpha_1-1}\left(\frac{k_1+t_1}{n_1}-x_1\right)^j dt_1 \\ &\quad \times\alpha_2 \sum_{k_2=0}^{\infty} l_{n_2,k_2}(x_2)\int_0^1 (1-t_2)^{\alpha_2-1}dt_2 \\ &= L_{n_1}^{\alpha_1}((e_1-x_1)^j;x_1). \end{aligned}$$

Similarly for $(e_{01}-x_2)^i$, we obtain

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{01}-x_2)^i;x_1,x_2) = L_{n_2}^{\alpha_2}((e_1-x_2)^i;x_2),$$

which completes the proof of the Lemma 5.3. □

Corollary 5.4. *Let $\alpha_1,\alpha_2 \in (0,\infty)$, $n_1,n_2 \in \mathbb{N}$ and $e_{ji}(t_1,t_2) = t_1^j t_2^i$, $0 \le j+i \le 2$, then for $(x_1,x_2) \in [0,\infty)\times[0,\infty)$, we have*

1. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{10}-x_1;x_1,x_2) = \frac{1}{n_1(\alpha_1+1)} = \zeta_{n_1,\alpha_1}(x_1)$;
2. $L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{01}-x_2;x_1,x_2) = \frac{1}{n_2(\alpha_2+1)} = \zeta_{n_2,\alpha_2}(x_2)$;
3. $L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{10}-x_1)^2;x_1,x_2) = \frac{2x_1}{n_1} + \frac{2}{n_1^2(\alpha_1+1)(\alpha_1+2)} = \beta_{n_1,\alpha_1}(x_1)$;
4. $L_{n_1,n_2}^{\alpha_1,\alpha_2}((e_{01}-x_2)^2;x_1,x_2) = \frac{2x_2}{n_2} + \frac{2}{n_2^2(\alpha_2+1)(\alpha_2+2)} = \beta_{n_2,\alpha_2}(x_2)$.

Proof. Using Lemma 5.3 and Lemma 1.6, we get

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{10}-x_1;x_1,x_2)=L^{\alpha_1}_{n_1}(e_1-x_1;x_1)=\frac{1}{n_1(\alpha_1+1)}.$$

Similarly,

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{01}-x_2;x_1,x_2)=\frac{1}{n_2(\alpha_2+1)}.$$

For $(e_{10}-x_1)^2$, we obtain

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{10}-x_1)^2;x_1,x_2)=L^{\alpha_1}_{n_1}((e_1-x_1)^2;x_1)=\frac{2x_1}{n_1}+\frac{2}{n_1^2(\alpha_1+1)(\alpha_1+2)}.$$

Similarly,

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{01}-x_2)^2;x_1,x_2)=\frac{2x_2}{n_2}+\frac{2}{n_2^2(\alpha_2+1)(\alpha_2+2)}.$$

This concludes the proof of Corollary 5.4. □

The modulus of continuity for $f\in C_B([0,\infty)\times[0,\infty))$ is defined as

$$w(f;\delta_1,\delta_2) = \sup\{|f(t_1,t_2)-f(x_1,x_2)| : (t_1,t_2),(x_1,x_2)\in[0,\infty)\times[0,\infty), |t_1-x_1|\le\delta_1, |t_2-x_2|\le\delta_2\}.$$

$w(f;\delta_1,\delta_2)$ satisfies the following properties:

1. $|f(t_1,t_2)-f(x_1,x_2)|\le w(f;\delta_1,\delta_2)\left(1+\frac{|t_1-x_1|}{\delta_1}\right)\left(1+\frac{|t_2-x_2|}{\delta_2}\right)$;
2. $w(f;\delta_1,\delta_2)\to 0$ as $\delta_1,\delta_2\to 0$.

Theorem 5.5. *Let $\alpha_1,\alpha_2\in(0,\infty)$, $n_1,n_2\in\mathbb{N}$ and $f\in C_B([0,\infty)\times[0,\infty))$, then for all $(x_1,x_2)\in[0,\infty)\times[0,\infty)$, we get*

$$|L^{\alpha_1,\alpha_2}_{n_1,n_2}(f(t_1,t_2)-f(x_1,x_2);x_1,x_2)|\le 4w\left(f;\sqrt{\beta_{n_1,\alpha_1}(x_1)},\sqrt{\beta_{n_2,\alpha_2}(x_2)}\right),$$

where $\beta_{n_1,\alpha_1}(x_1)$, $\beta_{n_2,\alpha_2}(x_2)$ are defined in Lemma 5.4.

Proof. Using property of modulus of continuity, we get

$$\begin{aligned}|L^{\alpha_1,\alpha_2}_{n_1,n_2}(f(t_1,t_2)-f(x_1,x_2);x_1,x_2)| &\le L^{\alpha_1,\alpha_2}_{n_1,n_2}(|f(t_1,t_2)-f(x_1,x_2)|;x_1,x_2)\\ &\le \left(1+\frac{L^{\alpha_1,\alpha_2}_{n_1,n_2}(|t_1-x_1|;x_1,x_2)}{\delta_1}\right)\left(1+\frac{L^{\alpha_1,\alpha_2}_{n_1,n_2}(|t_2-x_2|;x_1,x_2)}{\delta_2}\right)w(f;\delta_1,\delta_2).\end{aligned}$$

Using the Cauchy-Schwarz's inequality, we obtain

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}(|t_1-x_1|;x_1,x_2)\le(L^{\alpha_1,\alpha_2}_{n_1,n_2}((t_1-x_1)^2;x_1,x_2))^{\frac{1}{2}}(L^{\alpha_1,\alpha_2}_{n_1,n_2}(1;x_1,x_2))^{\frac{1}{2}}=\sqrt{\beta_{n_1,\alpha_1}(x_1)}.$$

Similarly,

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}(|t_2-x_2|;x_1,x_2)\le\sqrt{\beta_{n_2,\alpha_2}(x_2)}.$$

Hence

$$|L^{\alpha_1,\alpha_2}_{n_1,n_2}(f(t_1,t_2)-f(x_1,x_2);x_1,x_2)|\le\left(1+\frac{\sqrt{\beta_{n_1,\alpha_1}(x_1)}}{\delta_1}\right)\left(1+\frac{\sqrt{\beta_{n_2,\alpha_2}(x_2)}}{\delta_2}\right)w(f;\delta_1,\delta_2).$$

Taking $\delta_1=\sqrt{\beta_{n_1,\alpha_1}(x_1)}$ and $\delta_2=\sqrt{\beta_{n_2,\alpha_2}(x_2)}$, we have

$$|L^{\alpha_1,\alpha_2}_{n_1,n_2}(f(t_1,t_2)-f(x_1,x_2);x_1,x_2)|\le 4w\left(f;\sqrt{\beta_{n_1,\alpha_1}(x_1)},\sqrt{\beta_{n_2,\alpha_2}(x_2)}\right).$$

Thus the proof is completed. □

Lipschitz class $Lip_T(\lambda_1, \lambda_2)$ for $f \in C_B([0,\infty) \times [0,\infty))$ is defined as

$$Lip_T(\lambda_1, \lambda_2) = \{f(x_1, x_2) \in C_B([0,\infty) \times [0,\infty)) : |f(t_1, t_2) - f(x_1, x_2)| \leq T\|t_1 - x_1\|^{\lambda_1}\|t_2 - x_2\|^{\lambda_2}\},$$

where $(t_1, t_2), (x_1, x_2) \in [0,\infty) \times [0,\infty)$ and $T > 0$ is constant.

Theorem 5.6. *Let $\alpha_1, \alpha_2 \in (0,\infty)$, $n_1, n_2 \in \mathbb{N}$ and $f \in Lip_T(\lambda_1, \lambda_2)$, then for all $(x_1, x_2) \in [0,\infty) \times [0,\infty)$, we get*

$$\left|L_{n_1,n_2}^{\alpha_1,\alpha_2}(f(t_1, t_2) - f(x_1, x_2); x_1, x_2)\right| \leq T\left(\beta_{n_1,\alpha_1}(x_1)\right)^{\frac{\lambda_1}{2}}\left(\beta_{n_2,\alpha_2}(x_2)\right)^{\frac{\lambda_2}{2}}.$$

Proof. Using definition of Lipschitz class, we obtain

$$\begin{aligned}\left|L_{n_1,n_2}^{\alpha_1,\alpha_2}(f(t_1, t_2) - f(x_1, x_2); x_1, x_2)\right| &\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(|f(t_1, t_2) - f(x_1, x_2)|; x_1, x_2) \\ &\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(T|t_1 - x_1|^{\lambda_1}|t_2 - x_2|^{\lambda_2}; x_1, x_2) \\ &= TL_{n_1,n_2}^{\alpha_1,\alpha_2}(|t_1 - x_1|^{\lambda_1}; x_1, x_2)L_{n_1,n_2}^{\alpha_1,\alpha_2}(|t_2 - x_2|^{\lambda_2}; x_1, x_2). \qquad (11)\end{aligned}$$

Using the Hölder's inequality with $p_1 = \frac{2}{\lambda_1}$ and $q_1 = \frac{2}{2-\lambda_1}$, we get

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(|t_1 - x_1|^{\lambda_1}; x_1, x_2) \leq (L_{n_1,n_2}^{\alpha_1,\alpha_2}((t_1 - x_1)^2; x_1, x_2))^{\frac{1}{p_1}}(L_{n_1,n_2}^{\alpha_1,\alpha_2}(1; x_1, x_2))^{\frac{1}{q_1}} = (\beta_{n_1,\alpha_1}(x_1))^{\frac{\lambda_1}{2}}.$$

Similarly using the Hölder's inequality with $p_2 = \frac{2}{\lambda_2}$ and $q_2 = \frac{2}{2-\lambda_2}$, we have

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(|t_2 - x_2|^{\lambda_2}; x_1, x_2) \leq (\beta_{n_2,\alpha_2}(x_2))^{\frac{\lambda_2}{2}}.$$

Hence by inequality (11), we obtain

$$|L_{n_1,n_2}^{\alpha_1,\alpha_2}(f(t_1, t_2) - f(x_1, x_2); x_1, x_2)| \leq T(\beta_{n_1,\alpha_1}(x_1))^{\frac{\lambda_1}{2}}(\beta_{n_2,\alpha_2}(x_2))^{\frac{\lambda_2}{2}}.$$

Hence, the proof of Theorem 5.6 is now complete. □

Theorem 5.7. *Let $f \in C([0,\infty) \times [0,\infty))$, then sequence of operators $L_{n_1,n_2}^{\alpha_1,\alpha_2}(f)$ converges to f uniformly on $I \times J$, where I, J are compact intervals, subset of $[0,\infty)$.*

Proof. Using Lemma 5.2, we get

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{00}; x_1, x_2) = e_{00}.$$

For e_{10}, we have

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{10}; x_1, x_2) = \lim_{n_1,n_2\to\infty}\left(x_1 + \frac{1}{n_1(\alpha_1 + 1)}\right) = x_1.$$

Similarly,

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{01}; x_1, x_2) = e_{01}(x_1, x_2).$$

For e_{20}, we obtain

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{20}; x_1, x_2) = \lim_{n_1,n_2\to\infty}\left(x_1^2 + \frac{2(\alpha_1 + 2)x_1}{(\alpha_1 + 1)n_1} + \frac{2}{n_1^2(\alpha_1 + 1)(\alpha_1 + 2)}\right) = x_1^2.$$

Similarly,

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{02}; x_1, x_2) = e_{02}(x_1, x_2).$$

Using Linearity of $L_{n_1,n_2}^{\alpha_1,\alpha_2}$, we get

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{20} + e_{02}; x_1, x_2) = e_{20} + e_{02}.$$

Hence, by [8, theorem 2.1], $L_{n_1,n_2}^{\alpha_1,\alpha_2}(f)$ converges to f uniformly. □

Lemma 5.8. *Let $\alpha_1, \alpha_2 \in (0, \infty)$ and $n \in \mathbb{N}$, then for all $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, we obtain*

1. $\lim_{n\to\infty} nL_{n,n}^{\alpha_1,\alpha_2}(e_{10} - x_1; x_1, x_2) = \dfrac{1}{(\alpha_1 + 1)}$;
2. $\lim_{n\to\infty} nL_{n,n}^{\alpha_1,\alpha_2}(e_{01} - x_2; x_1, x_2) = \dfrac{1}{(\alpha_2 + 1)}$;
3. $\lim_{n\to\infty} nL_{n,n}^{\alpha_1,\alpha_2}((e_{10} - x_1)^2; x_1, x_2) = 2x_1$;
4. $\lim_{n\to\infty} nL_{n,n}^{\alpha_1,\alpha_2}((e_{01} - x_2)^2; x_1, x_2) = 2x_2$.

Proof. Using Corollary 5.4 and Lemma 1.8, one can obtain. □

Theorem 5.9. *Let $\alpha_1, \alpha_2 \in (0, \infty)$, $f \in C^2([0, \infty) \times [0, \infty))$ and $n \in \mathbb{N}$, then for all $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, we get*

$$\lim_{n\to\infty} n[L_{n,n}^{\alpha_1,\alpha_2}(f; x_1, x_2) - f(x_1, x_2)] = \frac{f_{x_1}(x_1, x_2)}{\alpha_1 + 1} + x_1 f_{x_1x_1}(x_1, x_2) + \frac{f_{x_2}(x_1, x_2)}{\alpha_2 + 1} + x_2 f_{x_2x_2}(x_1, x_2).$$

Proof. Using the Taylor's theorem on f, we have

$$\begin{aligned} f(t_1, t_2) &= f(x_1, x_2) + f_{x_1}(x_1, x_2)(t_1 - x_1) + f_{x_2}(x_1, x_2)(t_2 - x_2) + \rho(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4} \\ &\quad + \frac{1}{2}\left[f_{x_1x_1}(x_1, x_2)(t_1 - x_1)^2 + 2f_{x_1x_2}(x_1, x_2)(t_1 - x_1)(t_2 - x_2) + f_{x_2x_2}(x_1, x_2)(t_2 - x_2)^2\right], \end{aligned}$$

where $\rho(t_1, t_2; x_1, x_2) \in C([0, \infty) \times [0, \infty))$, $\lim_{(t_1,t_2)\to(x_1,x_2)} \rho(t_1, t_2; x_1, x_2) = 0$.

Applying $L_{n,n}^{\alpha_1,\alpha_2}$ and multiplying by n, we obtain

$$\begin{aligned} nL_{n,n}^{\alpha_1,\alpha_2}(f; x_1, x_2) &= nf(x_1, x_2) + nf_{x_1}(x_1, x_2)L_{n,n}^{\alpha_1,\alpha_2}(t_1 - x_1; x_1, x_2) + nf_{x_2}(x_1, x_2)L_{n,n}^{\alpha_1,\alpha_2}(t_2 - x_2; x_1, x_2) \\ &\quad + \frac{n}{2} f_{x_1x_1}(x_1, x_2)L_{n,n}^{\alpha_1,\alpha_2}((t_1 - x_1)^2; x_1, x_2) + \frac{n}{2} f_{x_2x_2}(x_1, x_2)L_{n,n}^{\alpha_1,\alpha_2}((t_2 - x_2)^2; x_1, x_2) \\ &\quad + nf_{x_1x_2}(x_1, x_2)L_{n,n}^{\alpha_1,\alpha_2}((t_1 - x_1)(t_2 - x_2); x_1, x_2) \\ &\quad + nL_{n,n}^{\alpha_1,\alpha_2}(\rho(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2). \end{aligned} \tag{12}$$

Using the Cauchy-Schwarz's inequality, we get

$$\begin{aligned} & L_{n,n}^{\alpha_1,\alpha_2}(\rho(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2) \\ & \le [L_{n,n}^{\alpha_1,\alpha_2}(\rho^2(t_1, t_2; x_1, x_2); x_1, x_2)]^{\frac{1}{2}}[L_{n,n}^{\alpha_1,\alpha_2}((t_1 - x_1)^4 + (t_2 - x_2)^4; x_1, x_2)]^{\frac{1}{2}}. \end{aligned}$$

Since $L_{n,n}^{\alpha_1,\alpha_2}(\rho^2(t_1, t_2; x_1, x_2); x_1, x_2) \to 0$ as $n \to \infty$ uniformly on $[0, \infty) \times [0, \infty)$, we get

$$\lim_{n\to\infty} nL_{n,n}^{\alpha_1,\alpha_2}(\rho(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2) = 0. \tag{13}$$

Taking limit on (12) and using corollary 5.4, lemma 5.8 and equation (13), we have

$$\lim_{n\to\infty} n[L_{n,n}^{\alpha_1,\alpha_2}(f; x_1, x_2) - f(x_1, x_2)] = \frac{f_{x_1}(x_1, x_2)}{\alpha_1 + 1} + x_1 f_{x_1x_1}(x_1, x_2) + \frac{f_{x_2}(x_1, x_2)}{\alpha_2 + 1} + x_2 f_{x_2x_2}(x_1, x_2).$$

Thus the proof is completed. □

Conclusion and Future direction

The Lupaş-Kantorovich operators are the integral modification of the Lupaş operators. In this paper, we expanded the classical integral modification to the Riemann-Liouville type integral dependent on the parameter $\alpha \in [0, \infty)$ for Lupaş operators. This generalization produces better error estimates than the classical case. We provided theoretical evidence for the approximation of these operators. We obtain the weighted approximation properties and Voronovskaja's theorem. In addition, we investigated the bivariate approximation properties of Riemann-Liouville type Lupaş-Kantorovich operators. It is possible to create Riemann-Liouville integral modifications for various positive linear operators like Jain operators, Meyer-König-Zeller operators, Post-Widder operators, modified Szász–Kantorovich operators, and many more. Other fractional integral like Caputo, Grünwald-Letnikov, Fourier-based, and convolution-based can be introduced to study the approximation properties of Bernstein-Kantorovich, Baskakov-Kantorovich, Szasz-Mirakjan-Kantorovich, Lupaş-Kantorovich and many more operators defined in approximation theory.

Ethics declarations

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by the authors.

Informed consent

For this type of study informed consent was not required.

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