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A unified approach to new discrete local fractional Hilbert-type inequalities

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Abstract. The main objective of this paper is a study of some new discrete local fractional Hilbert-type inequalities. We apply our general results to homogeneous kernels. Finally, the best possible constants are also obtained.

1. Introduction

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [9]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1}$$

where the constant π is the best possible. The inequality (1) is well known as Hilbert's integral inequality, which is important in mathematical analysis and its applications.

Over the past decade, fractional integral inequalities (FIIs) attracted widespread attention from an increasing number of scholars at home and abroad since they play a significant role in discussions of the quantitative and qualitative behavior of solutions to fractional differential equations. Currently, a large number of FIIs have been obtained, the reader can refer to related references [1–3, 5, 10–13, 21]. For example, by using one/two fractional parameters, Dahmani et al. [7] established some new generalisations of Grüss-type inequality for Riemann-Liouville fractional integral operators (RLFIOs). Based on the multiplicative fractional integral identity and multiplicative RLFIOs, some Hermite–Hadamard type inequalities for multiplicatively convex functions were established [8]. By using the RLFIOs, Sahoo et al. [15] presented the Ostrowski–Mercer inequalities and variants of Jensen's inequality for differentiable convex functions. By adopting the parametrized integral identity via Atangana–Baleanu FIOs, certain Simpson-like integral

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inequalities were obtained for mappings with (s, P)-convex and (s, P)-concave second-order derivatives in absolute value [24]. By employing two local fractional integral identities with the first- and second-order derivatives, Sun [17] considered some new Hermite-Hadamard–type local FIIs with Mittag-Leffler function (MLF) for generalized h-convex functions. Sarikaya and Budak [16] derived the generalized Ostrowski inequality and related inequalities using the generalized convex function for local FIOs on fractal sets. Based on the unified FIOs with an extended generalized MLF and Marichev-Saigo-Maeda FIOs, Yang [19, 20] studied some weighted Young, Shisha-Mond, Diaz-Metcalf, and Pólya-Szegö-type inequalities, respectively.

In this paper, motivated by mentioned results above, a new Hilbert-type inequalities are built by using local FIOs. First, we give basic definitions and results of the local fractional calculus (see [22, 23]). Let $\mathbb{R}$ be real numbers. There exist responding real line numbers on a fractal set E with fractal dimension α $(0 < \alpha \leq 1)$, denoted by $\mathbb{R}^\alpha$. We define the addition and multiplication operations on $\mathbb{R}^\alpha$ by $a^\alpha + b^\alpha := (a+b)^\alpha$ and $a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha$, $a^\alpha, b^\alpha \in \mathbb{R}^\alpha$. Obviously, with these two operations, $\mathbb{R}^\alpha$ is a field with an additive identity 0^α and a multiplicative identity 1^α.

Furthermore, we introduce the local fractional derivative and integral.

Definition 1. A non-differentiable function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$, there exists for $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon^\alpha$ holds for $0 < |x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_\alpha(a, b)$.

Definition 2. Let $f(x) \in C_\alpha[a, b]$. The local fractional derivative of the function $f(x)$ at $x = x_0$ is given by

$$f^{(\alpha)}(x_0) = {}_{x_0}D_x^\alpha f(x) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Gamma(1 + \alpha)$ stands for the classic gamma function. Assume $f^{(\alpha)}(x) = D_x^\alpha f(x)$. If there has $f^{(\varpi\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{\varpi \text{ times}} f(x)$ for any $f \in I \subseteq \mathbb{R}$, then we say that $f \in D_{\varpi\alpha}(I)$, where $\varpi = 1, 2, \ldots$.

Definition 3. Let $f(x) \in C_\alpha[a, b]$ and $P = \{t_0, t_1, \ldots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Furthermore, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \ldots, N - 1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$. Then the local FIO of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^\alpha f(x)$) is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.$$

The above definition implies that ${}_aI_b^{(\alpha)} f(x) = 0$ if $a = b$, and ${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{(\alpha)} f(x)$, then we denote by $f(x) \in I_x^{(\alpha)}[a, b]$.

At the end of this summary, we give some usefull formulas:

(a1) $\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}, \quad k > 0;$

(a2) $\frac{d^\alpha E_\alpha((cx)^\alpha)}{dx^\alpha} = c^\alpha E_\alpha((cx)^\alpha)$, where $E_\alpha(\cdot)$ denotes the MLF by $E_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$;

(a3) If $y(x) = (f \circ g)(x)$, then $\frac{d^\alpha y(x)}{dx^\alpha} = f^{(\alpha)}(g(x))(g'(x))^\alpha$;

(a4) $\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha)$;

(a5) $\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha} - a^{(k+1)\alpha}), \quad k > 0;$

(a6) $B_\alpha(a,b) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{x^{\alpha(b-1)}}{(1^\alpha + x^\alpha)^{a+b}} (dx)^\alpha$, where $B_\alpha(a,b)$ denotes local fractional Beta function.

Besides, we introduce the following notation and definition (see [6]).

Definition 4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. If the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) \tag{2}$$

holds for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, then f is said to be a generalized convex function on I.

In this paper, by using the way of weight functions and the technique of local fractional calculus, a new Hilbert-type discrete inequality with homogeneous kernel and a best constant is bulilt. As applications, the equivalent form and some particular cases are obtained.

2. Main results

The starting point in improvements to Hilbert-type inequalities is the well-known Hölder's inequality. A fractal version of Hölder's inequality is given in the following lemma (see also [18]).

Lemma 1. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let h, F, $G \in C_\alpha(\mathbb{R}_+^2)$ be non-negative functions. If*

$$0 < \sum_{m=1}^\infty \sum_{n=1}^\infty h(m,n) F^p(m,n) < \infty, \quad 0 < \sum_{m=1}^\infty \sum_{n=1}^\infty h(m,n) G^q(m,n) < \infty,$$

then the following inequality holds

$$\sum_{m=1}^\infty \sum_{n=1}^\infty h(m,n) F(m,n) G(m,n) \le \Big(\sum_{m=1}^\infty \sum_{n=1}^\infty h(m,n) F^p(m,n) \Big)^{\frac{1}{p}} \Big(\sum_{m=1}^\infty \sum_{n=1}^\infty h(m,n) G^q(m,n) \Big)^{\frac{1}{p}}. \tag{3}$$

The preceding lemma will help us to prove our main result.

Theorem 1. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be non-negative real sequences. If φ, $\psi \in C_\alpha(\mathbb{R}_+)$ and $K \in C_\alpha(\mathbb{R}_+)^2$ is non-negative decreasing function in both variables on $\mathbb{R}_+$, then the following inequalities hold and are equivalent*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) a_m^\alpha b_n^\alpha \le \Big(\sum_{m=1}^\infty (\varphi F)^p(m) a_m^{\alpha p} \Big)^{\frac{1}{p}} \Big(\sum_{n=1}^\infty (\psi G)^q(n) b_n^{\alpha q} \Big)^{\frac{1}{q}}, \tag{4}$$

$$\Big(\sum_{n=1}^\infty (\psi G)^{-p}(n) \Big(\sum_{m=1}^\infty K(m,n) a_m^\alpha \Big)^p \Big)^{\frac{1}{p}} \le \Big(\sum_{m=1}^\infty (\varphi F)^p(m) a_m^{\alpha p} \Big)^{\frac{1}{p}}, \tag{5}$$

where

$$F^p(m) := \sum_{n=1}^\infty K(m,n) \psi^{-p}(n) \ \ \text{and} \ \ G^q(n) := \sum_{m=1}^\infty K(m,n) \varphi^{-q}(m). \tag{6}$$

Proof. The left-hand side of inequality (4) can be written in the following form

$$\sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) a_m^\alpha b_n^\alpha = \sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) a_m^\alpha \frac{\varphi(m)}{\psi(n)} b_n^\alpha \frac{\psi(n)}{\varphi(m)}.$$

Now, applying the fractal Hölder's inequality (3) to the above identity yields

$$\sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) a_m^\alpha b_n^\alpha \le \Big(\sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) a_m^\alpha \frac{\varphi^p(m)}{\psi^p(n)} \Big)^{\frac{1}{p}} \Big(\sum_{m=1}^\infty \sum_{n=1}^\infty K(m,n) b_n^\alpha \frac{\psi^q(n)}{\varphi^q(m)} \Big)^{\frac{1}{q}}.$$

Finally, using the local fractional Fubini theorem and definitions of functions F and G we obtain (4).

Now, we are going to prove the equivalence of inequalities (4) and (5). For this sake, suppose that inequality (4) holds. Define the following sequence $(b_n)_{n\in\mathbb{N}}$ by

$$b_n^\alpha = (G\psi)^{-p}(n)\Big(\sum_{m=1}^{\infty} K(m,n)a_m^\alpha\Big)^{p-1}$$

and from the inequality (4), we have

$$\begin{aligned}\sum_{n=1}^{\infty}(\psi G)^{-p}(n)\Big(\sum_{m=1}^{\infty} K(m,n)a_m^\alpha\Big)^{p} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m^\alpha b_n^\alpha \\ &\le \Big(\sum_{m=1}^{\infty}(\varphi F)^p(m)a_m^{\alpha p}\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}(\psi G)^q(n)b_n^{\alpha q}\Big)^{\frac{1}{q}} \\ &= \Big(\sum_{m=1}^{\infty}(\varphi F)^p(m)a_m^{\alpha p}\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}(\psi G)^{-p}(n)\Big(\sum_{m=1}^{\infty} K(m,n)a_m^\alpha\Big)^{p}\Big)^{\frac{1}{q}},\end{aligned}$$

that is, we get (5).

On the other hand, suppose that inequality (5) holds. In that case, another use of the fractal Hölder's inequality (3) yields

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m^\alpha b_n^\alpha &= \sum_{n=1}^{\infty}\Big((\psi G)^{-1}(n)K(m,n)a_m^{\alpha p}\Big)(\psi G)(n)b_n^{\alpha q} \\ &\le\Big(\sum_{n=1}^{\infty}(\psi G)^{-p}(n)\Big(\sum_{m=1}^{\infty} K(m,n)a_m^\alpha\Big)^{p}\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}(\psi G)^q(n)b_n^{\alpha q}\Big)^{\frac{1}{q}} \\ &\le\Big(\sum_{m=1}^{\infty}(\varphi F)^p(m)a_m^{\alpha p}\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}(\psi G)^q(n)b_n^{\alpha q}\Big)^{\frac{1}{q}},\end{aligned}$$

which implies (4). Hence, inequalities (4) and (5) are equivalent. □

In what follows. we suppose that $K \in C_\alpha(\mathbb{R}_+^2)$ is a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. Furthermore, we define the integral $k(\cdot)$ by

$$k(\eta) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(1,t)t^{-\alpha\eta}(dt)^\alpha, \tag{7}$$

under assumption $k(\eta) < \infty$.

Besides, we study below discrete weight functions involving real differentiable functions. More precisely, we are introducing the following notation and definition.

Definition 5. Let $r > 0$. We denote by $H(r)$ the set of all non-negative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(i) u is increasing on $\mathbb{R}_+$ and holds $0 < u(1) < \infty$;

(ii) $\lim\limits_{x\to\infty} u(x) = \infty$, $\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$ is decreasing and generalized convex function on $\mathbb{R}_+$.

By applying local fractional calculus we can easily get the next lemma.

Lemma 2. *Let $r > 0$ and $K(x, y)$ be decreasing and generalized convex function in both variables on $\mathbb{R}_+$. If $u, v \in H(r)$, then*

$$K(u(x), v(y))\frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}} \quad \text{and} \quad K(u(x), v(y))\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$$

are decreasing and generalized convex function on $\mathbb{R}_+$ for any $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_+$, respectively.

Proof. For the sake of proof, we set $V(y) := \frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}}$, and suppose that $x \in \mathbb{R}_+$. Then

$$\frac{\partial^\alpha}{\partial y^\alpha}[K(u(x), v(y))V(y)] = \frac{\partial^\alpha}{\partial y^\alpha}[K(u(x), v(y))] \cdot V(y) + K(u(x), v(y))\frac{\partial^\alpha}{\partial y^\alpha}[V(y)] \le 0$$

and similarly

$$\begin{aligned}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}[K(u(x), v(y))V(y)] =& \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}[K(u(x), v(y))] \cdot V(y) + \frac{\partial^\alpha}{\partial y^\alpha}[K(u(x), v(y))]\frac{\partial^\alpha}{\partial y^\alpha}[V(y)] \\ &+ \frac{\partial^\alpha}{\partial y^\alpha}[K(u(x), v(y))]\frac{\partial^\alpha}{\partial y^\alpha}[V(y)] + K(u(x), v(y))\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}[V(y)] \ge 0.\end{aligned}$$

In the same way follows proof for the function $K(u(x), v(y))\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$. □

We need the next technical lemma. (see [18]).

Lemma 3. *If $f \in I_x^{(\alpha)}(\mathbb{R}_+)$, $f^{(\alpha)}(t) \le 0$, $f^{(2\alpha)}(t) \ge 0$ $(t \in (1/2, \infty))$, then we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_1^\infty f(t)(dt)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^\infty f(n) \le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^\infty f(t)(dt)^\alpha. \tag{8}$$

Now, we will prove the result with a general homogeneous kernel that has some properties.

Theorem 2. *Let $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be non-negative real sequences and $K \in C_\alpha(\mathbb{R}_+^2)$ be a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. If $u \in H(pA_2)$, $v \in H(qA_1)$ and K is decreasing and generalized convex function in both variables on $\mathbb{R}_+$, then the following inequalities hold and are equivalent*

$$\begin{aligned}\sum_{m=1}^\infty\sum_{n=1}^\infty K(u(m), v(n))a_m^\alpha b_n^\alpha \le& L\Big(\sum_{m=1}^\infty [u(m)]^{\alpha(1-s)+\alpha p(A_1-A_2)}[u'(m)]^{\alpha(1-p)}a_m^{\alpha p}\Big)^{\frac{1}{p}} \\ &\times\Big(\sum_{n=1}^\infty [v(n)]^{\alpha(1-s)+\alpha q(A_2-A_1)}[v'(n)]^{\alpha(1-q)}b_n^{\alpha q}\Big)^{\frac{1}{q}},\end{aligned} \tag{9}$$

$$\begin{aligned}\Big(\sum_{n=1}^\infty [v(n)]^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}[v'(n)]^\alpha\Big(\sum_{m=1}^\infty K(u(m), v(n))a_m^\alpha\Big)^p\Big)^{\frac{1}{p}} \\ \le L\Big(\sum_{m=1}^\infty [u(m)]^{\alpha(1-s)+\alpha p(A_1-A_2)}[u'(m)]^{\alpha(1-p)}a_m^{\alpha p}\Big)^{\frac{1}{p}},\end{aligned} \tag{10}$$

where $A_1 \in (\max\{(1-s)/q, 0\}, 1/q)$, $A_2 \in (\max\{(1-s)/p, 0\}, 1/p)$ and

$$L = \Gamma(1+\alpha)k(pA_2)^{\frac{1}{p}}k(2-s-qA_1)^{\frac{1}{q}}. \tag{11}$$

Proof. We put the functions $(\varphi\circ u)(m)=[u(m)]^{\alpha A_1}[u'(m)]^{-\frac{\alpha}{q}}$, $(\psi\circ v)(n)=[v(n)]^{\alpha A_2}[v'(n)]^{-\frac{\alpha}{p}}$ in inequality (4). These substitutions are well defined, since u and v are injective functions. Thus, in this setting we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))a_m^{\alpha}b_n^{\alpha}\leq\left(\sum_{m=1}^{\infty}[u(m)]^{\alpha pA_1}[u'(m)]^{\alpha(1-p)}(F\circ u)(m)a_m^{\alpha p}\right)^{\frac{1}{p}}\times\left(\sum_{n=1}^{\infty}[v(n)]^{\alpha qA_2}[v'(n)]^{\alpha(1-q)}(G\circ v)(n)b_n^{\alpha q}\right)^{\frac{1}{q}},\quad(12)$$

where

$$(F\circ u)(m)=\sum_{n=1}^{\infty}\frac{K(u(m),v(n))[v'(n)]^{\alpha}}{[v(n)]^{\alpha pA_2}}\quad\text{and}\quad(G\circ v)(n)=\sum_{m=1}^{\infty}\frac{K(u(m),v(n))[u'(m)]^{\alpha}}{[u(m)]^{\alpha qA_1}}.$$

Note that $v\in H(pA_2)$ and K is decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Hence, applying Lemma 2 and Lemma 3, we obtain

$$(F\circ u)^p(m)\leq\Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\frac{K(u(m),v(y))}{[v(y)]^{\alpha pA_2}}[v'(y)]^{\alpha}(dy)^{\alpha}.\quad(13)$$

Furthermore, by using substitution $t=\frac{v(y)}{u(m)}$ and homogeneity of function K, we have

$$(F\circ u)^p(m)\leq\Gamma(1+\alpha)[u(m)]^{\alpha-\alpha s-\alpha pA_2}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(1,t)t^{-\alpha pA_2}(dt)^{\alpha},$$

so we get by (7)

$$(F\circ u)^p(m)\leq\Gamma(1+\alpha)[u(m)]^{\alpha-\alpha s-\alpha pA_2}k(pA_2).\quad(14)$$

By the similar arguments as for function $F\circ u$, we get

$$\begin{aligned}(G\circ v)(m)&\leq\Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\frac{K(u(x),v(n))}{[u(x)]^{\alpha qA_1}}[u'(x)]^{\alpha}(dx)^{\alpha}\\&\leq\Gamma(1+\alpha)[v(n)]^{\alpha-\alpha s-\alpha qA_1}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(t,1)t^{-\alpha qA_1}(dt)^{\alpha}\\&=\Gamma(1+\alpha)[v(n)]^{\alpha-\alpha s-\alpha qA_1}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(1,t)t^{\alpha s+\alpha qA_1-2\alpha}(dt)^{\alpha}\\&=\Gamma(1+\alpha)[v(n)]^{\alpha-\alpha s-\alpha qA_1}k(2-s-qA_1).\end{aligned}\quad(15)$$

Finally, relations (12), (14), and (15) yield the inequality (9).

On the other hand, if we rewrite inequality (5) with the same functions as in the proof of inequality (9), after using estimates (14) and (15), we easily get (10). That completes the proof. □

The main idea in obtaining the best possible constant factor is a reduction of constant L defined by (11) in the form without exponents. It is natural to install the next condition

$$pA_2+qA_1=2-s\quad(16)$$

that the relation $k(pA_2)=k(2-s-qA_1)$ holds. On that way, the constant L from Theorem 2 becomes

$$L^*=\Gamma(1+\alpha)k(pA_2).\quad(17)$$

Further, under assumption (16), the inequalities (9) and (10) respectively can be rewritten as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))a_m^{\alpha}b_n^{\alpha}\leq L^*\Big(\sum_{m=1}^{\infty}[u(m)]^{-\alpha+\alpha pqA_1}[u'(m)]^{\alpha(1-p)}a_m^{\alpha p}\Big)^{\frac{1}{p}}$$
$$\times\Big(\sum_{n=1}^{\infty}[v(n)]^{-\alpha+\alpha pqA_2}[v'(n)]^{\alpha(1-q)}b_n^{\alpha q}\Big)^{\frac{1}{q}},\quad (18)$$

$$\Big(\sum_{n=1}^{\infty}[v(n)]^{\alpha(p-1)(1-pqA_2)}[v'(n)]^{\alpha}\Big(\sum_{m=1}^{\infty}K(u(m),v(n))a_m^{\alpha}\Big)^{p}\Big)^{\frac{1}{p}}\leq L^*\Big(\sum_{m=1}^{\infty}[u(m)]^{-\alpha+\alpha pqA_1}[u'(m)]^{\alpha(1-p)}a_m^{\alpha p}\Big)^{\frac{1}{p}}.\quad (19)$$

Our main aim is to show that the constants involved in the right-hand sides of inequalities (18) and (19) are the best possible. That is, we present the content of the following theorem.

Theorem 3. *Let s, A_1, A_2, u, v and $K(x,y)$ be defined as in Theorem 2. If the parameters A_1 and A_2 satisfy condition* (16)*, then the constant factor L^* is the best possible in inequalities* (18) *and* (19)*.*

Proof. It is enough to show that the constant L^* is the best possible in inequality (18), since (18) and (19) are equivalent inequalities.

For this purpose, put $\widetilde{a}_m^{\alpha}=[u(m)]^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}[u'(m)]^{\alpha}$ and $\widetilde{b}_n^{\alpha}=[v(n)]^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}s[v'(n)]^{\alpha}$, where $0<\varepsilon q<1-pA_2$. Let us suppose that the inequality (18) holds if we put the sequences $(\widetilde{a}_m)$ and $(\widetilde{b}_n)$. By using Lemma 3 (see also [18]), we have

$$\begin{aligned}\frac{1}{\varepsilon^{\alpha}[u(1)]^{\alpha}\Gamma(1+\alpha)}&=\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}[u(x)]^{-\alpha-\varepsilon}[u'(x)]^{\alpha}(dx)^{\alpha}\leq\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{\infty}[u(m)]^{-\alpha-\alpha\varepsilon}[u'(m)]^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{\infty}[u(m)]^{-\alpha+\alpha pqA_1}[u'(m)]^{\alpha(1-p)}\widetilde{a}_m^{\alpha p}\\&\leq\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}[u(x)]^{-\alpha-\alpha\varepsilon}[u'(x)]^{\alpha}(dx)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}[u(x)]^{-\alpha-\alpha\varepsilon}[u'(x)]^{\alpha}(dx)^{\alpha}.\end{aligned}$$

Hence, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{\infty}[u(m)]^{-\alpha+\alpha pqA_1}[u'(m)]^{\alpha(1-p)}\widetilde{a}_m^{\alpha p}=\frac{1}{\varepsilon^{\alpha}[u(1)]^{\alpha}\Gamma(1+\alpha)}+O(1),\quad (20)$$

and similarly

$$\frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}[v(n)]^{-\alpha+\alpha pqA_2}[v'(n)]^{\alpha(1-q)}\widetilde{b}_n^{\alpha q}=\frac{1}{\varepsilon^{\alpha}[v(1)]^{\alpha}\Gamma(1+\alpha)}+O(1).\quad (21)$$

Now, let us suppose that there exists a positive constant $M<L^*$ such that the inequality (18) is still valid, if we replace L^* with M. Without loss of generality, we suppose that $u(1)\leq v(1)$. Hence, if we insert relations (20) and (21) in inequality (18), with the constant M instead of L^*, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m^{\alpha}\widetilde{b}_n^{\alpha}\leq\frac{1}{\varepsilon^{\alpha}[v(1)]^{\alpha}\Gamma(1+\alpha)}(M+o(1)).\quad (22)$$

On the other hand, we estimate the left-hand side of inequality (18). If we insert the above defined sequences $(\widetilde{a}_m)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n)_{n\in\mathbb{N}}$ in the left-hand side of (18), we get the inequality

$$J_{\varepsilon}:=\frac{1}{\Gamma^2(1+\alpha)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m^{\alpha}\widetilde{b}_n^{\alpha}\geq\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}[u(x)]^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}[u'(x)]^{\alpha}$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}K(u(x),v(y))[v(y)]^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}[v'(y)]^{\alpha}(dy)^{\alpha}\right)(dx)^{\alpha}, \quad (23)$$

where we use Lemma 2 and Lemma 3. By using the substitution $t = v(y)/u(x)$ we have

$$J_\varepsilon \geq \frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}[u(x)]^{-\alpha-\alpha\varepsilon}[u'(x)]^{\alpha}\left(\int_{\frac{v(1)}{u(x)}}^{\infty}K(1,t)t^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dt)^{\alpha}\right)(dx)^{\alpha}. \quad (24)$$

Furthermore, since the kernel K is strictly decreasing in both arguments, it follows that $K(1,0) > K(1,t)$, for $t > 0$, so we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_{\frac{v(1)}{u(x)}}^{\infty}K(1,t)t^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dt)^{\alpha}\\ &\quad\geq \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(1,t)t^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dt)^{\alpha}-\frac{K(1,0)}{\Gamma(1+\alpha)}\int_0^{\frac{v(1)}{u(x)}}t^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dt)^{\alpha}\\ &\quad= k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{K(1,0)}{\Gamma(1+\alpha)(1-pA_2-\frac{\varepsilon}{q})^{\alpha}}\left(\frac{u(x)}{v(1)}\right)^{\alpha pA_2+\frac{\alpha\varepsilon}{q}-\alpha},\end{aligned}$$

and consequently

$$J_\varepsilon \geq \frac{k\left(pA_2+\frac{\varepsilon}{q}\right)}{\varepsilon^{\alpha}[u(1)]^{\alpha}\Gamma(1+\alpha)}+\frac{K(1,0)}{\Gamma^2(1+\alpha)[v(1)]^{\alpha pA_2+\frac{\alpha\varepsilon}{q}-\alpha}}\frac{1}{\left(1-pA_2-\frac{\varepsilon}{q}\right)^{\alpha}\left(1-pA_2+\frac{\varepsilon}{p}\right)^{\alpha}}. \quad (25)$$

In other words, relations (23), (24) and (25) yield the estimate for the left-hand side of inequality (18):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m^{\alpha}\widetilde{b}_n^{\alpha} \geq \frac{1}{\varepsilon^{\alpha}[u(1)]^{\alpha\varepsilon}}(L^*+o(1)). \quad (26)$$

Finally, by comparing relations (22) and (26), and by letting $\varepsilon \to 0^+$, we conclude that $L^* \leq M$, which contradicts with the assumption that the constant M is smaller than L^*. That means that L^* is the best possible constant in inequality (18).

The equivalence of the inequalities (18) and (19) means that the constant L^* is the best possible in the inequality (19). The proof is completed now. □

We proceed with some special homogeneous functions. First, we prove that the kernel $K_1(x,y) = (x^{\alpha}+y^{\alpha})^{-s}$, $s > 0$, is decreasing and generalized convex function in both variables on $\mathbb{R}_+$. By using local fractional calculus, we have

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{1}{(m+x)^{\alpha s}} = -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{1}{(m+x)^{\alpha(s+1)}} < 0, \quad x > 0,$$

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{1}{(m+x)^{\alpha s}} = \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{1}{(m+x)^{\alpha(s+2)}} > 0, \quad x > 0.$$

Since the function K_1 is homogeneous of degree $-\alpha s$, by using Lemma 2 and Theorem 3 we obtain the following result.

Corollary 1. *Let $s, A_1, A_2, u(x), v(y)$ and $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be defined as in Theorem 2. Suppose that the parameters A_1, A_2 satsify condition* (16). *Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^{\alpha}b_n^{\alpha}}{(u(m)+v(n))^{\alpha s}} \leq M\left(\sum_{m=1}^{\infty}[u(m)]^{-\alpha+\alpha pqA_1}[u'(m)]^{\alpha(1-p)}a_m^{\alpha p}\right)^{\frac{1}{p}}$$

$$\times \Big(\sum_{n=1}^{\infty} [v(n)]^{-\alpha+\alpha pqA_2} [v'(n)]^{\alpha(1-q)} b_n^{\alpha q} \Big)^{\frac{1}{q}}, \quad (27)$$

$$\Big(\sum_{n=1}^{\infty} [v(n)]^{\alpha(p-1)(1-pqA_2)} [v'(n)]^{\alpha} \Big(\sum_{m=1}^{\infty} \frac{a_m}{(u(m)+v(n))^{\alpha s}} \Big)^p \Big)^{\frac{1}{p}} \le M \Big(\sum_{m=1}^{\infty} [u(m)]^{-\alpha+\alpha pqA_1} [u'(m)]^{\alpha(1-p)} a_m^{\alpha p} \Big)^{\frac{1}{p}}, \quad (28)$$

where the constant $M = \Gamma(1+\alpha) B_\alpha(1-pA_2, 1-qA_1)$ *is the best possible.*

In what follows, we suppose that

$$A_1 = \frac{2-s}{2q}, \quad A_2 = \frac{2-s}{2p}. \quad (29)$$

Moreover, setting $u(x) = Ax^c$, $v(y) = By^d$, $A, B, c, d > 0$ and the parameters defined by (29), we obtain the following result.

Corollary 2. *Let* $0 < s < 2$, $0 < c, d < 2/s$, $A, B > 0$. *Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m^{\alpha} b_n^{\alpha}}{(Am^c + Bn^d)^{\alpha s}} \le M_1 \Big(\sum_{m=1}^{\infty} m^{\alpha p - \alpha + \frac{\alpha c p s}{2}} a_m^{\alpha p} \Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} n^{\alpha q - \alpha + \frac{\alpha d q s}{2}} b_n^{\alpha q} \Big)^{\frac{1}{q}}, \quad (30)$$

$$\Big(\sum_{n=1}^{\infty} n^{\frac{\alpha c p s}{2} + \alpha d - \alpha} \Big(\sum_{m=1}^{\infty} \frac{a_m^{\alpha}}{(Am^c + Bn^d)^{\alpha s}} \Big)^p \Big)^{\frac{1}{p}} \le M_1 \Big(\sum_{m=1}^{\infty} m^{\alpha p - \alpha + \frac{\alpha c p s}{2}} a_m^{\alpha p} \Big)^{\frac{1}{p}}, \quad (31)$$

where the constant

$$M_1 = A^{-\frac{\alpha s}{2}} B^{-\frac{\alpha s}{2}} c^{-\frac{\alpha}{q}} d^{-\frac{\alpha}{q}} \Gamma(1+\alpha) B_\alpha \left(\frac{s}{2}, \frac{s}{2} \right)$$

is the best possible.

Remark 1. If we put $u(x) = v(x) = x + \mu$, $\mu > 0$, $A_1 = A_2 = (2-s)/(pq)$, $0 < s < 2$, in Corollary 1, then the inequalities (27) and (28) become

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m^{\alpha} b_n^{\alpha}}{(m+n+2\mu)^{\alpha s}} \le M_2 \Big(\sum_{m=1}^{\infty} (m+\mu)^{\alpha - \alpha s} a_m^{\alpha p} \Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} (n+\mu)^{\alpha - \alpha s} b_n^{\alpha q} \Big)^{\frac{1}{q}},$$

$$\Big(\sum_{n=1}^{\infty} (n+\mu)^{\alpha(p-1)(s-1)} \Big(\sum_{m=1}^{\infty} \frac{a_m^{\alpha}}{(m+n+2\mu)^{\alpha s}} \Big)^p \Big)^{\frac{1}{p}} \le M_2 \Big(\sum_{m=1}^{\infty} (m+\mu)^{\alpha - \alpha s} a_m^{\alpha p} \Big)^{\frac{1}{p}},$$

where the constant

$$M_2 = \Gamma(1+\alpha) B_\alpha \Big(\frac{1}{p} + \frac{s-1}{q}, \frac{1}{q} + \frac{s-1}{p} \Big)$$

is the best possible. Here we obtain a case with non-homogeneous kernel in Corollary 2. For $s = 1$, we get non-weighted case with the best possible constant $M_2 = \Gamma(1+\alpha) B_\alpha(1/p, 1/q)$.

If we put $u(x) = \kappa v(x) + \mu$, $\kappa, \mu > 0$ in Corollary 1, then the inequalities (27) and (28) become

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m^{\alpha} b_n^{\alpha}}{(\kappa v(m) + v(n) + \mu)^{\alpha s}} \le M_3 \Big(\sum_{m=1}^{\infty} [\kappa v(m) + \mu]^{-\alpha + \alpha pqA_1} [v'(m)]^{\alpha(1-p)} a_m^{\alpha p} \Big)^{\frac{1}{p}}$$

$$\times \Big(\sum_{n=1}^{\infty} [v(n)]^{-\alpha+\alpha pqA_2} [v'(n)]^{\alpha(1-q)} b_n^{\alpha q} \Big)^{\frac{1}{q}},$$

$$\Big(\sum_{n=1}^{\infty} [v(n)]^{\alpha(p-1)(1-pqA_2)} [v'(n)]^{\alpha} \Big(\sum_{m=1}^{\infty} \frac{a_m}{(\kappa v(m) + v(n) + \mu)^{\alpha s}} \Big)^p \Big)^{\frac{1}{p}}$$

$$\le M_3 \Big(\sum_{m=1}^{\infty} [\kappa v(m) + \mu]^{-\alpha + \alpha pqA_1} [v'(m)]^{\alpha(1-p)} a_m^{\alpha p} \Big)^{\frac{1}{p}},$$

where the constant $M_3 = \Gamma(1+\alpha) \kappa^{-\frac{\alpha}{q}} B_\alpha(1-pA_2, 1-qA_1)$ is the best possible.

3. Conclusion

Based on the fractal Hölder's inequality, we have obtained some new discrete local fractional Hilbert-type inequalities. As applications, some special homogeneous functions have been used as the kernel functions. We have also proved that the constant factors in the desired inequalities are the best possible.

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