Filomat 39:9 (2025), 3201–3212
https://doi.org/10.2298/FIL2509201L

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some approximation results for the Bézier variants of two generalized Bernstein type operators

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Abstract. In this paper we introduce two new Bézier variants of the generalized Bernstein type operators. With the help of Ditzian Totik smooth modulus, we discuss a global approximation. Also, we obtain the rate of convergence of the operators by the second order continuous modulus and Peetre K-functional. By using the construction of appropriate functions and the methods of Bojanic-Cheng, the approximation properties of the new defined operators for absolute continuous functions with derivatives equivalent to bounded variation functions is given. Finally, we provide four graphs to illustrate the approximation effect of the newly defined operators.

1. Introduction

Very recently, Usta [22] introduced a new family of generalized Bernstein operators as follows:

$$ß_n(\nu, x) = \sum_{k=0}^{n} \nu(\frac{k}{n}) p^*_{n,k}(x), \quad x \in [0,1], \nu \in C[0,1], \tag{1.1}$$

where $p^*_{n,k}(x) = \frac{1}{n}\binom{n}{k}(k-nx)^2 x^{k-1}(1-x)^{n-k-1}$.

In [22], the author studied the approximation properties of the operators $ß_n(\nu, x)$, such as asymptotic formulas, weighted approximations and convergence rates. In addition, numerical simulations were also included.

To approximate Lebesgue integrable functions, Senapati et al. [20] introduced the following integral modification of the operators (1.1):

$$\aleph_n(\nu, x) = (n+1)\sum_{k=0}^{n} p^*_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \nu(t)dt. \tag{1.2}$$

2020 *Mathematics Subject Classification*. Primary 41A36; Secondary 41A25, 41A35.
Keywords. Bernstein operators, modulus of smoothness, rate of convergence, bounded variation.
Received: 25 April 2024; Accepted: 20 January 2025
Communicated by Miodrag Spalević
Research supported by the Natural Science Foundation of Fujian Province of China (Grant No. 2023J011434) and the Discipline leader training programs of Yang-en University (Grant No. 20211006).
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In [20], the paper established the approximation properties of $\aleph_n(\nu, x)$, such as weighted approximation, local and global approximation degrees and Voronovskaya type theorem. Finally, the authors provided some numerical experiments to verify the theoretical results.

Base on the new defined operators (1.1), Kajla [13] and Liu [18] studied the Durrmeyer variant of $ß_n(\nu, x)$ and the blending-type Bernstein-Durrmeyer operators, respectively. Cai [7] dealt with the new modification of the Bernstein-Beta operators, preserving constant and Korovkin's other test functions in limit case. Sofyalıoğlu [21] defined the parametric generalization of the operators $ß_n(\nu, x)$ according to the idea of Chen [8].

As is well known, Bézier curves have extensive applications in computer-aided geometric design and computer graphics. Bézier variant operators play an important role in approximation theory. They can be used to construct new approximation operators that have better approximation properties on specific function classes. For example, by introducing Bézier variants, the performance of traditional operators in approximating bounded variation functions can be improved. The Bézier variant operators can also be used for numerical integration and differentiation. By utilizing the properties of these operators, more efficient numerical methods can be designed for calculating the integral and derivative of functions. In addition, Bézier curves have wide application value in curve and surface design, animation and simulation, signal and processing, and so on. The research on Bézier type operators has never been interrupted [1,5,11,14-17,19,23].

In this paper, we introduce the Bézier variant of operators (1.1) and (1.2) in the following way:

$$ß_{n,\alpha}(\nu, x) = \sum_{k=0}^{n} \nu(\frac{k}{n})\Theta_{n,k}^{(\alpha)}(x), \quad x \in [0, 1], \tag{1.3}$$

$$\aleph_{n,\alpha}(\nu, x) = (n+1)\sum_{k=0}^{n} \Theta_{n,k}^{(\alpha)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \nu(t)dt, \quad x \in [0, 1], \tag{1.4}$$

where $\alpha \geq 1$, $\Theta_{n,k}^{(\alpha)}(x) = [\Upsilon_{n,k}(x)]^{\alpha} - [\Upsilon_{n,k+1}(x)]^{\alpha}$, $\Upsilon_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}^{*}(x)$ and $\Upsilon_{n,n+1}(x) = 0$.

As everyone knows that for $0 \leq m, n \leq 1$ and $\alpha \geq 1$, the inequality $|m^{\alpha} - n^{\alpha}| \leq \alpha|m - n|$ holds. So we have

$$\Theta_{n,k}^{(\alpha)}(x) \leq \alpha p_{n,k}^{*}(x). \tag{1.5}$$

Obviously for $\alpha = 1$, operators (1.3) and (1.4) reduce to operators (1.1) and (1.2), respectively.

The purpose of this article is to establish a direct approximation using the global approximation theorem of Ditzian-Totik smoothness and second order continuous modulus. In addition, the convergence rate of absolute continuous functions with derivatives equivalent to bounded variation are also obtained. Finally, we provide some approximation graph examples of the newly defined operators. We can refer to the literatures [2-4,12] for research in this area.

2. Some lemmas

The proof of our results are based on the following lemmas.

Lemma 2.1 ([22]) *For $x \in [0, 1]$, we have*

$$ß_n(1, x) = 1, \tag{2.1}$$

$$ß_n(t, x) = x + \frac{1-2x}{n}, \tag{2.2}$$

$$ß_n(t^2, x) = x^2 + \frac{(6-7n)x^2 + (5n-6)x + 1}{n^2}, \tag{2.3}$$

$$ß_n(t-x,x) = \frac{1-2x}{n}, \tag{2.4}$$

$$ß_n((t-x)^2,x) = \frac{(6-3n)x^2+(3n-6)x+1}{n^2} = \Delta_n^2(x). \tag{2.5}$$

By Lemma 2.1 and Cauchy Schwarz inequality, we get

$$ß_n(|t-x|,x) \leq \sqrt{ß_n((t-x)^2,x)}\cdot\sqrt{ß_n(1,x)} = \Delta_n(x). \tag{2.6}$$

Lemma 2.2 ([20]) *For $x\in[0,1]$, we have*

$$\aleph_n(1,x) = 1, \tag{2.7}$$

$$\aleph_n(t,x) = x + \frac{3(1-2x)}{2(n+1)}, \tag{2.8}$$

$$\aleph_n(t^2,x) = x^2 + \frac{3(5-9n)x^2+3(6n-8)x+7}{3(n+1)^2}, \tag{2.9}$$

$$\aleph_n(t-x,x) = \frac{3(1-2x)}{2(n+1)}, \tag{2.10}$$

$$\aleph_n((t-x)^2,x) = \frac{3(11-3n)x^2+3(3n-11)x+7}{n^2} = \Lambda_n^2(x). \tag{2.11}$$

By Lemma 2.2 and Cauchy Schwarz inequality, we get

$$\aleph_n(|t-x|,x) \leq \sqrt{\aleph_n((t-x)^2,x)}\cdot\sqrt{\aleph_n(1,x)} = \Lambda_n(x). \tag{2.12}$$

Lemma 2.3 *For $\nu\in C[0,1]$, $x\in[0,1]$, we have*

$$||ß_{n,\alpha}(\nu,x)|| \leq \alpha||\nu||$$

and

$$||\aleph_{n,\alpha}(\nu,x)|| \leq \alpha||\nu||.$$

Proof According to the definition of (1.3) and the inequality of (1.5), we get

$$||ß_{n,\alpha}(\nu,x)|| \leq \sum_{k=0}^{n} |\,\nu(\frac{k}{n})\,|\,\Theta_{n,k}^{(\alpha)}(x) \leq \alpha\sum_{k=0}^{n} |\,\nu(\frac{k}{n})\,|\,p_{n,k}^{*}(x) \leq \alpha||\nu||\sum_{k=0}^{n} p_{n,k}^{*}(x) = \alpha||\nu||.$$

The inequality $||\aleph_{n,\alpha}(\nu,x)|| \leq \alpha||\nu||$ also follows the same path of proof.

3. Local Approximation

Lemma 3.1 ([9]) *For* $\nu(x) \in C[0,1]$ and $r > 0$, there exists an absolute constant $D > 0$ such that

$$K_2(\nu, r) \le D\omega_2(\nu, \sqrt{r}). \tag{3.1}$$

where $W^2[0,1] = \{g \in C[0,1] : g'' \in C[0,1]\}$, the Peetre K-functional $K_2(\nu, r)$ and the second order modulus of continuity $\omega_2(\nu, \sqrt{r})$ are defined as follows:

$$K_2(\nu, r) = \inf_{g \in W^2[0,1]} \{\|\nu - g\| + r\|g'\| + r^2\|g''\|\},$$

$$\omega_2(\nu, \sqrt{r}) = \sup_{0<|h|\le\sqrt{r}} \sup_{x, x+h, x+2h \in [0,1]} |\nu(x+2h) - 2\nu(x+h) + \nu(x)|.$$

Theorem 3.1 *For* $\nu \in C[0,1]$ *and* $x \in [0,1]$, *we have*

$$|ß_{n,\alpha}(\nu, x) - \nu(x)| \le D\omega_2\left(\nu, \sqrt[4]{\frac{\alpha\Delta_n^2(x)}{4}}\right), \tag{3.2}$$

where D is a positive constant.

Proof Let $g \in W^2$. By Taylor's expansion, we can write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying $ß_{n,\alpha}(\cdot, x)$ to both sides of the above equation, we have

$$ß_{n,\alpha}(g, x) = g(x) + g'(x)ß_{n,\alpha}(t-x, x) + ß_{n,\alpha}\left(\int_x^t (t-u)g''(u)du, x\right).$$

Noting $ß_{n,\alpha}(1, x) = 1$, we get

$$\begin{aligned}
|ß_{n,\alpha}(g,x) - g(x)| &\le |g'(x)|\left|ß_{n,\alpha}(|t-x|, x)\right| + \left|ß_{n,\alpha}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\le \|g'\|ß_{n,\alpha}(|t-x|, x) + \frac{\|g''\|}{2}ß_{n,\alpha}\left((t-x)^2, x\right) \\
&\le \|g'\|\left[ß_{n,\alpha}\left((t-x)^2, x\right)\right]^{1/2} + \frac{\|g''\|}{2}ß_{n,\alpha}\left((t-x)^2, x\right) \\
&\le \sqrt{\alpha}\|g'\|\left[ß_n\left((t-x)^2, x\right)\right]^{1/2} + \alpha\frac{\|g''\|}{2}ß_n\left((t-x)^2, x\right) \\
&= \sqrt{\alpha}\|g'\|\Delta_n(x) + \alpha\frac{\|g''\|}{2}\Delta_n^2(x).
\end{aligned}$$

So

$$\begin{aligned}
|ß_{n,\alpha}(\nu,x) - \nu(x)| &\le |ß_{n,\alpha}(\nu - g, x)| + |\nu - g| + |ß_{n,\alpha}(g,x) - g(x)| \\
&\le 2\|\nu - g\| + \sqrt{\alpha}\|g'\|\Delta_n(x) + \alpha\frac{\|g''\|}{2}\Delta_n^2(x).
\end{aligned}$$

Taking the infimum on the right hand side of above inequality for all $g \in W^2$, we obtain

$$|ß_{n,\alpha}(\nu, x) - \nu(x)| \le 2K_2\left(\nu, \sqrt{\frac{\alpha\Delta_n^2(x)}{4}}\right).$$

Using Lemma 3.1 we obtain Theorem 3.1 immediately.

By using a completely similar proof method, the following conclusion can be drawn.

Theorem 3.2 *For $\nu \in C[0,1]$ and $x \in [0,1]$, we have*

$$|\aleph_{n,\alpha}(\nu,x) - \nu(x)| \leq D\omega_2\left(\nu, \sqrt[4]{\frac{\alpha\Lambda_n^2(x)}{4}}\right), \tag{3.3}$$

where D is a positive constant.

As we know, a function ν belongs to the Lipschitz class $Lip_M(\beta)(0 < \beta \leq 1, M > 0)$ if the inequality

$$|\nu(t) - \nu(x)| \leq M|t-x|^\beta$$

holds for all $t, x \in R$. Now we compute the rate of convergence of the operators $ß_{n,\alpha}(\nu,x)$ and $\aleph_{n,\alpha}(\nu,x)$ for the Lipschitz class functions.

Theorem 3.3 *For $x \in [0,1]$ and $\nu \in Lip_M(\beta) \bigcap C[0,1]$, we have*

$$\left|ß_{n,\alpha}(\nu,x) - \nu(x)\right| \leq \alpha M\left[\Delta_n(x)\right]^\beta. \tag{3.4}$$

Proof Applying the Hölder inequality with $p = \frac{2}{\beta}, q = \frac{2}{2-\beta}$ and Lemma 2.3, we get

$$\begin{aligned}\left|ß_{n,\alpha}(\nu,x) - \nu(x)\right| &\leq ß_{n,\alpha}\left(|\nu(t)-\nu(x)|, x\right) \leq \alpha \cdot ß_n\left(|\nu(t)-\nu(x)|, x\right) \leq \alpha \cdot M \cdot ß_{n,\alpha}\left(|t-x|^\beta, x\right) \\ &\leq \alpha \cdot M \cdot \left[ß_n((t-x)^2, x)\right]^{\beta/2} \cdot \left[ß_n(1,x)\right]^{(2-\beta)/2} = \alpha \cdot M \cdot \left[\Delta_n(x)\right]^\beta.\end{aligned}$$

The last equation is obtained by (2.1) and (2.5).

By applying the same proof method, we can obtain the following conclusion.

Theorem 3.4 *For $x \in [0,1]$ and $\nu \in Lip_M(\beta) \bigcap C[0,1]$, we have*

$$\left|\aleph_{n,\alpha}(\nu,x) - \nu(x)\right| \leq \alpha M\left[\Lambda_n(x)\right]^\beta. \tag{3.5}$$

4. Global Approximation

Lemma 4.1 ([10]) *For $\varsigma(x) = \sqrt{x(1-x)}, s > 0$* and $\nu \in C[0,1]$, there exists a constant $E > 0$ such that

$$K_\varsigma(\nu,s) \leq E\omega_\varsigma(\nu,s). \tag{4.1}$$

Here $W_\varsigma[0,1] = \{g : g \in AC[0,1], \|\varsigma g'\| < \infty\}$ means that g is differentiable and absolutely continuous on the interval $[0,1]$, the first order Ditzian-Totik modulus of smoothness and corresponding K-functional are given by

$$\omega_\varsigma(\nu,s) = \sup_{0<\tau\leq s}\left|\nu(x + \frac{\tau\varsigma(x)}{2}) - \nu(x - \frac{\tau\varsigma(x)}{2})\right|, x \pm \frac{\tau\varsigma(x)}{2} \in [0,1]$$

and

$$K_\varsigma(\nu,s) = \inf_{g\in W_\varsigma[0,1]}\{\|\nu - g\| + s\|\varsigma g'\|\}(s > 0),$$

respectively.

Theorem 4.1 *For $\nu \in C[0,1]$, $x \in (0,1)$ and $\varsigma(x) = \sqrt{x(1-x)}$, we have*

$$|ß_{n,\alpha}(\nu,x) - \nu(x)| \leq E\omega_\varsigma\left(\nu, \sqrt{\frac{2\alpha}{\varsigma(x)}}\Delta_n(x)\right), \tag{4.2}$$

where E is a positive constant.

Proof Using the representation $g(t) = g(x) + \int_x^t g'(z)dz$ and $ß_{n,\alpha}(1, x) = 1$, we get

$$ß_{n,\alpha}(g, x) = g(x) + ß_{n,\alpha}\left(\int_x^t g'(z)dz, x\right). \tag{4.3}$$

For any $x, t \in (0, 1)$, we can get

$$\left|\int_x^t g'(z)dz\right| \leq \|\varsigma g'\| \left|\int_x^t \frac{1}{\varsigma(z)}dz\right|. \tag{4.4}$$

Next, we estimate this item $\int_x^t \frac{1}{\varsigma(z)}dz$.

$$\begin{aligned}\left|\int_x^t \frac{1}{\varsigma(z)}dz\right| &= \left|\int_x^t \frac{1}{\sqrt{z(1-z)}}dz\right| \leq \left|\int_x^t \left(\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{1-z}}\right)dz\right| \\ &= 2\left(|\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}|\right) \\ &= 2|t-x|\left(\frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}}\right) \\ &\leq 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}|t-x|}{\varsigma(x)}.\end{aligned}$$

From (4.3),(4.4),(2.5) and the Cauchy Schwarz inequality, we get

$$\begin{aligned}|ß_{n,\alpha}(g, x) - g(x)| &\leq 2\sqrt{2}\|\varsigma g'\|\varsigma^{-1}(x)ß_{n,\alpha}(|t-x|, x) \\ &\leq 2\sqrt{2}\|\varsigma g'\|\varsigma^{-1}(x)\left(ß_{n,\alpha}((t-x)^2, x)\right)^{1/2} \\ &\leq 2\sqrt{2}\sqrt{\alpha}\|\varsigma g'\|\varsigma^{-1}(x)\left(ß_n((t-x)^2, x)\right)^{1/2} \\ &= 2\sqrt{2}\sqrt{\alpha}\|\varsigma g'\|\varsigma^{-1}(x)\Delta_n(x).\end{aligned}$$

The reciprocal first inequality is obtained from Lemma 2.3.

So

$$\begin{aligned}|ß_{n,\alpha}(\nu, x) - \nu(x)| &\leq |ß_{n,\alpha}(\nu - g, x)| + |\nu - g| + |ß_{n,\alpha}(g, x) - g(x)| \\ &\leq 2\|\nu - g\| + 2\sqrt{2}\sqrt{\alpha}\|\varsigma g'\|\varsigma^{-1}(x)\Delta_n(x).\end{aligned}$$

Taking the infimum on the right hand side of above inequality for all $g \in W_\varsigma[0, 1]$, we obtain

$$|ß_{n,\alpha}(\nu, x) - \nu(x)| \leq 2K_\varsigma\left(\nu, \sqrt{\frac{2\alpha}{\varsigma(x)}}\Delta_n(x)\right).$$

Using Lemma 4.1 and the above inequality we get Theorem 4.1 immediately.

By using a completely similar proof method, the following conclusion can be drawn.

Theorem 4.2 *For $\nu \in C[0, 1]$, $x \in (0, 1)$ and $\varsigma(x) = \sqrt{x(1-x)}$, we have*

$$|\aleph_{n,\alpha}(\nu, x) - \nu(x)| \leq E\omega_\phi\left(f, \sqrt{\frac{2\alpha}{\varsigma(x)}}\Lambda_n(x)\right), \tag{4.5}$$

where E is a positive constant.

5. Rate of Convergence

In this section, we study the approximation properties of $ß_{n,\alpha}(\nu, x)$ and $\aleph_{n,\alpha}(\nu, x)$ for functions with a derivative of bounded variation on $[0, 1]$.

Let $\partial_{BV}[0, 1]$ denote the class of absolutely continuous functions defined on $[0, 1]$, whose derivatives have bounded variation on $[0, 1]$. It is well known that the functions $\nu \in \partial_{BV}[0, 1]$ possess a representation

$$\nu(x) = \nu(0) + \int_0^x h(t)dt,$$

where $h \in BV[0, 1]$, i.e., h is a function of bounded variation on $[0, 1]$.

Let kernel functions

$$\varpi^{(1)}_{n,\alpha}(x, t) = \begin{cases} \sum_{k\le nt} \Theta^{(\alpha)}_{n,k}(x), & 0 < t \le 1; \\ 0, & t = 0. \end{cases}$$

and

$$\varpi^{(2)}_{n,\alpha}(x, t) = \sum_{k=0}^{n} (n + 1)\Theta^{(\alpha)}_{n,k}(x)\chi_k(t),$$

where $\chi_k(t)$ is the characteristic function of the interval $[\frac{k}{n+1}, \frac{k+1}{n+1}]$ with respect to $I = [0, 1]$. An integral expression form of the operators $ß_{n,\alpha}(\nu, x)$ and $\aleph_{n,\alpha}(\nu, x)$ can be given as follows:

$$ß_{n,\alpha}(\nu, x) = \int_0^1 \nu(t)d_t\varpi^{(1)}_{n,\alpha}(x, t), \tag{5.1}$$

$$\aleph_{n,\alpha}(\nu, x) = \int_0^1 \nu(t)\varpi^{(2)}_{n,\alpha}(x, t)dt. \tag{5.2}$$

Lemma 5.1 (i) *For $0 \le y < x < 1$, we have*

$$\varpi^{(1)}_{n,\alpha}(x, y) \le \frac{\alpha}{(x - y)^2}\Delta^2_n(x). \tag{5.3}$$

(ii) *For $0 < x < z \le 1$, we have*

$$1 - \varpi^{(1)}_{n,\alpha}(x, t) \le \frac{\alpha}{(x - y)^2}\Delta^2_n(x). \tag{5.4}$$

Proof (i) By the inequality of (1.5) and the expression of (5.1), we get

$$\begin{aligned} \varpi^{(1)}_{n,\alpha}(x, y) &\le \alpha\varpi^{(1)}_{n,1}(x, y) = \alpha \int_0^y d_t\varpi^{(1)}_{n,1}(x, t) \le \alpha \int_0^y \left(\frac{x - t}{x - y}\right)^2 d_t\varpi^{(1)}_{n,1}(x, t) \\ &\le \frac{\alpha}{(x - y)^2} \int_0^1 (t - x)^2 d_t\varpi^{(1)}_{n,1}(x, t) = \frac{\alpha}{(x - y)^2} ß_n((t - x)^2, x) \\ &= \frac{\alpha}{(x - y)^2}\Delta^2_n(x). \end{aligned}$$

(ii) Using a similar method we can get (5.4) easily.

Using the same proof method as Lemma 5.1, we get

Lemma 5.2 (i) *For $0 \le y < x < 1$, we have*

$$\int_0^y \varpi^{(2)}_{n,\alpha}(x, t)dt \le \frac{\alpha}{(x - y)^2}\Lambda^2_n(x). \tag{5.5}$$

(ii) *For $0 < x < z \leq 1$, we have*

$$\int_z^1 \varpi_{n,\alpha}^{(2)}(x,t)dt \leq \frac{\alpha}{(x-z)^2}\Lambda_n^2(x). \tag{5.6}$$

Theorem 5.1 *Let $\nu \in \partial_{BV}[0,1]$. Then, for every $x \in (0,1)$, the following inequality*

$$\begin{aligned}\left|ß_{n,\alpha}(\nu,x)-\nu(x)\right| &\leq \alpha\Big(|\nu'(x+)|+|\nu'(x-)|\Big)\Delta_n(x)+\frac{2\alpha\Delta_n^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(\varphi_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(\varphi_x)\\ &+\frac{2\alpha\Delta_n^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(\varphi_x)+\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x)\end{aligned}$$

holds, where

$$\varphi_x(t)=\begin{cases}\nu'(t)-\nu'(x+), & x<t\leq 1;\\ 0, & t=x;\\ \nu'(t)-\nu'(x-), & 0\leq t<x.\end{cases}$$

Proof For $\nu \in \partial_{BV}[0,1]$, we have $\nu(t)-\nu(x)=\int_x^t \nu'(u)du$. Using Bojanic-Cheng's decomposition [6], $\nu'(u)$ can be expressed as

$$\begin{aligned}\nu'(u) &= \varphi_x(u)+\frac{1}{2}\left[\nu'(x+)+\nu'(x-)\right]+\frac{1}{2}\left[\nu'(x+)-\nu'(x-)\right]sign(u-x)\\ &+\delta_x(u)\left[\nu'(x)-\frac{1}{2}\left(\nu'(x+)+\nu'(x-)\right)\right],\end{aligned} \tag{5.7}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x;\\ 0, & u\neq x.\end{cases}$$

$$sign(x)=\begin{cases}1, & x>0;\\ 0, & x=0;\\ -1, & x<0.\end{cases}$$

Noting $\int_x^t sign(u-x)du=|t-x|$, $\int_x^t \delta_x(u)du=0$ and $ß_{n,\alpha}(1,x)=1$, we get

$$\begin{aligned}\left|ß_{n,\alpha}(\nu,x)-\nu(x)\right| &= \left|ß_{n,\alpha}(\nu(t)-\nu(x),x)\right|=\left|ß_{n,\alpha}(\nu,x)(\int_x^t \nu'(u)du,x)\right|\\ &= \Big|\frac{\nu'(x+)+\nu'(x-)}{2}ß_{n,\alpha}(t-x,x)\\ &\quad+\frac{\nu'(x+)-\nu'(x-)}{2}ß_{n,\alpha}(|t-x|,x)+ß_{n,\alpha}(\int_x^t\varphi_x(u)du,x)\Big|\\ &\leq \left(|\nu'(x+)|+|\nu'(x-)|\right)ß_{n,\alpha}(|t-x|,x)+\left|ß_{n,\alpha}(\int_x^t\varphi_x(u)du,x)\right|.\end{aligned}$$

By Lemma 2.3 and (2.6), we get $ß_{n,\alpha}(|t-x|,x)\leq \alpha ß_n(|t-x|,x)\leq \alpha\Delta_n(x)$.
So

$$\left|ß_{n,\alpha}(\nu,x)-\nu(x)\right|\leq \alpha\left(|\nu'(x+)|+|\nu'(x-)|\right)\Delta_n(x)+\left|ß_{n,\alpha}(\int_x^t\varphi_x(u)du,x)\right|. \tag{5.8}$$

Thus, our task is to estimate the term $ß_{n,\alpha}(\int_x^t\varphi_x(u)du,x)$.

By the representations of (5.1), we can write

$$ß_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right) = \int_0^1 (\int_x^t \varphi_x(u)du)d_t\varpi_{n,\alpha}^{(1)}(x,t) = \Sigma_1 + \Sigma_2, \tag{5.9}$$

where $\Sigma_1 = \int_0^x(\int_x^t \varphi_x(u)du)d_t\varpi_{n,\alpha}^{(1)}(x,t)$ and $\Sigma_2 = \int_x^1(\int_x^t \varphi_x(u)du)d_t\varpi_{n,\alpha}^{(1)}(x,t)$.

Applying the integration by parts and noticing $\varpi_{n,\alpha}^{(1)}(x,0) = 0$, $\int_x^x \varphi_x(u)du = 0$, we get

$$\begin{aligned}\Sigma_1 &= \varpi_{n,\alpha}^{(1)}(x,t)\int_x^t \varphi_x(u)du\Big|_0^x - \int_0^x \varpi_{n,\alpha}^{(1)}(x,t)\varphi_x(t)dt \\ &= -\int_0^x \varpi_{n,\alpha}^{(1)}(x,t)\varphi_x(t)dt = -(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^x)\varpi_{n,\alpha}^{(1)}(x,t)\varphi_x(t)dt.\end{aligned}$$

Thus, it follows that

$$\left|\Sigma_1\right| \le \int_0^{x-\frac{x}{\sqrt{n}}} \varpi_{n,\alpha}^{(1)}(x,t)\bigvee_t^x(\varphi_x)dt + \int_{x-\frac{x}{\sqrt{n}}}^x \varpi_{n,\alpha}^{(1)}(x,t)\bigvee_t^x(\varphi_x)dt.$$

By $0 \le \varpi_{n,\alpha}^{(1)}(x,t) \le 1$ and (5.3), we get

$$\left|\Sigma_1\right| \le \alpha\Delta_n^2(x)\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{5.10}$$

By considering $t = x - \frac{x}{u}$, we yield

$$\left|\Sigma_1\right| \le \frac{\alpha\Delta_n^2(x)}{x}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x(\varphi_x)du + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{5.11}$$

According to (5.4) and the same method, we have

$$\left|\Sigma_2\right| \le \frac{\alpha\Delta_n^2(x)}{1-x}\int_1^{\sqrt{n}} \bigvee_x^{x+\frac{1-x}{u}}(\varphi_x)du + \frac{1-x}{\sqrt{n}}\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x). \tag{5.12}$$

By (5.9), (5.11) and (5.12), we get

$$\begin{aligned}ß_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right) &\le \frac{2\alpha\Delta_n^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x) \\ &+ \frac{2\alpha\Delta_n^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{1-x}{k}}(\varphi_x) + \frac{1-x}{\sqrt{n}}\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x).\end{aligned} \tag{5.13}$$

Theorem 5.1 now follows from (5.8) and (5.13) immediately.

Using the same proof method as Theorem 5.1, we get

Theorem 5.2 *Let $\nu \in \partial_{BV}[0,1]$. Then, for every $x \in (0,1)$, the following inequality*

$$\begin{aligned}\left|\aleph_{n,\alpha}(\nu,x) - \nu(x)\right| &\le \alpha\Big(|\nu'(x+)| + |\nu'(x-)|\Big)\Lambda_n(x) + \frac{2\alpha\Lambda_n^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x) \\ &+ \frac{2\alpha\Lambda_n^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{1-x}{k}}(\varphi_x) + \frac{1-x}{\sqrt{n}}\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x).\end{aligned}$$

holds.

6. Graphical analysis

In this section, we show several graphics to present the convergence of operators $\ss_{n,\alpha}(\nu, x)$ and $\aleph_{n,\alpha}(\nu, x)$ to certain functions with different values of n and α.

Example 1 Let $f(x) = e^{-x}sin(\frac{3\pi x}{2}) + 1$, $\alpha = 2$, and $n = 20, 30, 50, 100$.

Figure 1 shows the convergence of the operators $\ss_{n,\alpha}(\nu, x)$ to the function $f(x)$.

Obviously, as the random n increases, the error of the approximation $|\ss_{n,\alpha}(\nu, x) - \nu(x)|$ becomes smaller and smaller.

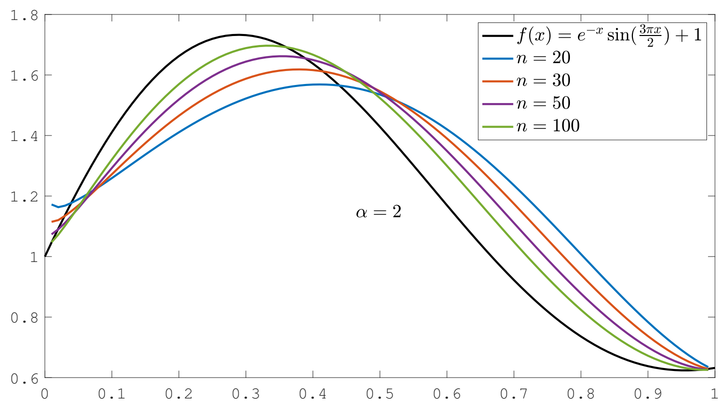

Figure 1: Convergence behaviour of $\ss_{n,\alpha}(\nu, x)$.

Example 2 Let $f(x) = e^{-x}sin(\frac{3\pi x}{2}) + 1$, $n = 50$, and $\alpha = 1, 2, 3, 4$.

Figure 2 shows the convergence of the operators $\ss_{n,\alpha}(\nu, x)$ to the function $f(x)$.

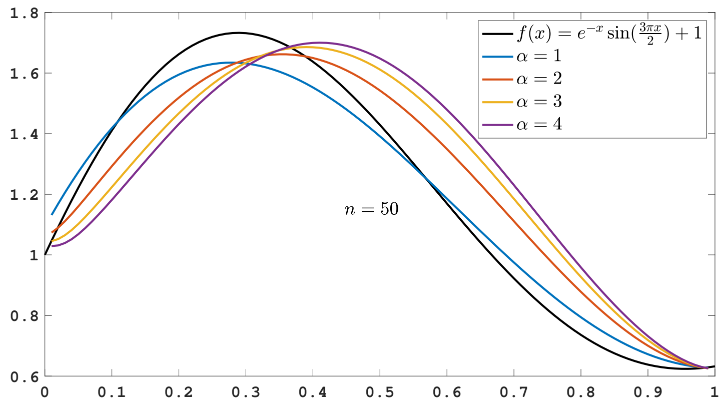

Figure 2: Convergence behaviour of $\ss_{n,\alpha}(\nu, x)$.

Example 3 Let $f(x) = sin(10x)e^{-3x} + 1$, $\alpha = 2$, and $n = 20, 30, 50, 100$.
Figure 3 shows the convergence of the operators $\aleph_{n,\alpha}(\nu, x)$ to the function $f(x)$.
Obviously, as the random n increases, the error of the approximation $|\aleph_{n,\alpha}(\nu, x) - \nu(x)|$ becomes smaller and smaller.

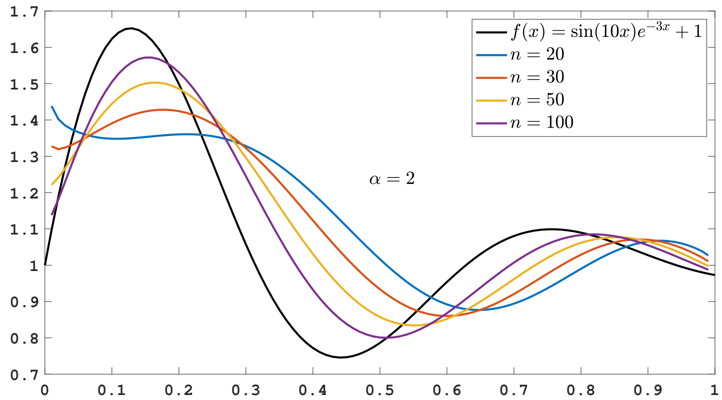

Figure 3: Convergence behaviour of $\aleph_{n,\alpha}(\nu, x)$.

Example 4 Let $f(x) = sin(10x)e^{-3x} + 1$, $n = 50$, and $\alpha = 1, 2, 3, 4$.
Figure 4 shows the convergence of the operators $\aleph_{n,\alpha}(\nu, x)$ to the function $f(x)$.

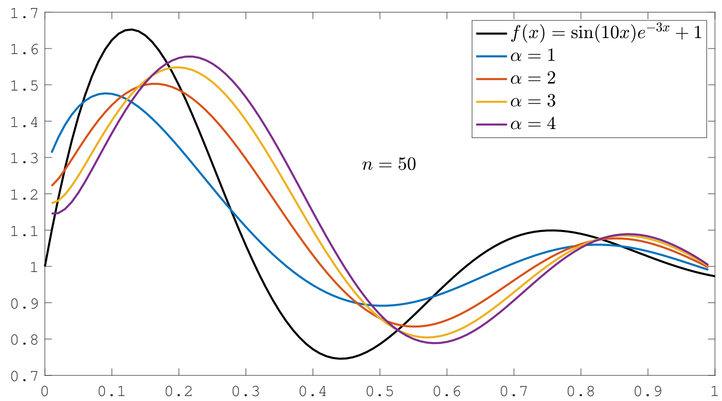

Figure 4: Convergence behaviour of $\aleph_{n,\alpha}(\nu, x)$.

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