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The largest α-sepctral radius of k-uniform bicyclic hypergraphs

Lou-Jun Yu[a], Wen-Huan Wang[a,b,*]

[a]*Department of Mathematics, Shanghai University, Shanghai 200444, China*
[b]*Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, China*

Abstract. Let $\mathcal{G}$ be a k-uniform hypergraph and $\mathcal{A}_\alpha(\mathcal{G}) = \alpha\mathcal{D}(\mathcal{G})+(1-\alpha)\mathcal{A}(\mathcal{G})$ the convex linear combination of its degree diagonal tensor $\mathcal{D}(\mathcal{G})$ and its adjacency tensor $\mathcal{A}(\mathcal{G})$, where $k \geq 3$ and $0 \leq \alpha < 1$. The α-spectral radius of $\mathcal{G}$ is the largest modulus of all the eigenvalues of $\mathcal{A}_\alpha(\mathcal{G})$. Let $\mathcal{B}(n, k)$ be the set of the connected k-uniform bicyclic hypergraphs, where $k \geq 3$. The number of the edges of the hypergraphs in $\mathcal{B}(n, k)$ is denoted by $m = \frac{n+1}{k-1}$. We develop a new ρ_α-normal labeling method for calculating the α-spectral radius of k-uniform hypergraphs. By using some transformations and the new ρ_α-normal labeling methods, we characterize the hypergraphs with the first and the second largest α-spectral radii among $\mathcal{B}(n, k)$, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

1. Introduction

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a hypergraph, where $V(\mathcal{G}) = \{v_1, \dots, v_n\}$ and $E(\mathcal{G}) = \{e_1, \dots, e_m\}$ are respectively the sets of the vertices and the edges of $\mathcal{G}$. For each edge $e \in E(\mathcal{G})$, if $|e| = k$, then $\mathcal{G}$ is a k-uniform hypergraph, where $k \geq 2$. In $\mathcal{G}$, a path of length p from v_1 to v_{p+1} is an alternating sequence $v_1e_1v_2 \dots v_pe_pv_{p+1}$ of vertices and edges such that $v_i, v_{i+1} \subseteq e_i$ for $i = 1, \dots, p$. A hypergraph is connected if there is a path connecting any two vertices of $\mathcal{G}$. For a k-uniform hypergraph $\mathcal{G}$, let $\omega(\mathcal{G})$ and $r(\mathcal{G})$ be its numbers of components and cyclomatics, respectively. A k-uniform hypergraph $\mathcal{G}$ is called $r(\mathcal{G})$-cyclic if $m(k-1) - n + \omega(\mathcal{G}) = r(\mathcal{G})$ holds [4]. If $\omega(\mathcal{G}) = 1$, then $\mathcal{G}$ is a connected hypergraph. If $r(\mathcal{G}) = 0, 1, 2$, then $\mathcal{G}$ is respectively a supertree, a k-uniform unicyclic hypergraph and a k-uniform bicyclic hypergraph. Thus, for a k-uniform bicyclic hypergraph $\mathcal{G}$, we have $m = \frac{n+1}{k-1}$. For a vertex $v \in V(\mathcal{G})$, the degree of v, denoted by d_v, is the number of the edges of $\mathcal{G}$ which are incident with v. A vertex of degree one is called a core vertex. A vertex of degree at least two is referred to as an intersection vertex (abbreviated as IV). A pendent edge means that it has only one IV. A non-pendent edge has at least two IVs.

A real tensor $\mathcal{A} = (a_{i_1i_2\cdots i_k}) \in \mathbb{R}^{n\times n\times\cdots\times n}$ of order k and dimension n over the real field $\mathbb{R}$ is a multi-dimensional array with n^k entries, where $a_{i_1i_2\cdots i_k} \in \mathbb{R}$ with $i_1, i_2, \cdots, i_k \in [n] = \{1, 2, \cdots, n\}$. In 2005, Qi [18] and Lim [11] independently introduced the concept of tensor eigenvalues and the spectra of tensors as

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* Corresponding author: Wen-Huan Wang
Email address: whwang@shu.edu.cn (Wen-Huan Wang)
ORCID iD: https://orcid.org/0009-0000-6478-4703 (Lou-Jun Yu), https://orcid.org/0000-0002-9292-5638 (Wen-Huan Wang)

follows. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ be an n-dimensional complex column vector, where $\mathbb{C}$ is the set of complex numbers. Let $\boldsymbol{x}^{[k]} = (x_1^k, x_2^k, \cdots, x_n^k)^{\mathrm{T}}$, where k is a positive integer. By using the product of tensors defined by Shao [21], $\mathcal{A}\boldsymbol{x}^{k-1}$ is simplified as $\mathcal{A}\boldsymbol{x}$. Then $\mathcal{A}\boldsymbol{x}$ is a vector in $\mathbb{C}^n$ whose i-th component is given by

$$(\mathcal{A}\boldsymbol{x}^{k-1})_i = (\mathcal{A}\boldsymbol{x})_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k}, \text{ for each } i \in [n]. \tag{1}$$

We have

$$\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}\boldsymbol{x}) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} \cdots x_{i_k}. \tag{2}$$

If there exist a number $\lambda \in \mathbb{C}$ and a nonzero eigenvector $\boldsymbol{x} \in \mathbb{C}^n$ such that $\mathcal{A}\boldsymbol{x}^{k-1} = \lambda \boldsymbol{x}^{[k-1]}$, namely $(\mathcal{A}\boldsymbol{x}^{k-1})_i = \lambda x_i^{k-1}$ for any $i \in [n]$, then $\boldsymbol{x}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ.

Let $\mathcal{G}$ be a k-uniform hypergraph with n vertices. In 2012, Cooper and Dutle [2] defined that the adjacency tensor of $\mathcal{G}$ is the k-ordered and n-dimensional tensor $\mathcal{A}(\mathcal{G}) = (a_{i_1 i_2 \cdots i_k})$, where $a_{i_1 i_2 \cdots i_k} = \frac{1}{(k-1)!}$ if $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(\mathcal{G})$ and $a_{i_1 i_2 \cdots i_k} = 0$ otherwise. Let $\mathcal{D}(\mathcal{G}) = (d_{i_1 i_2 \ldots i_k})$ be the degree diagonal tensor of order k and dimension n for $\mathcal{G}$, where $d_{i_1 i_2 \ldots i_k} = d_{v_i}$ if $i_1 = i_2 = \ldots = i_k = i$ with $v_i \in V(\mathcal{G})$ and $i = 1, \cdots, n$, and $d_{i_1 i_2 \ldots i_k} = 0$ otherwise with $i_1, i_2, \ldots, i_k \in [n]$. In 2017, Nikiforov [15] proposed to merge the spectral properties of the adjacency matrix and the signless Laplacian matrix of a graph. Let $\mathcal{A}_\alpha(\mathcal{G}) = \alpha \mathcal{D}(\mathcal{G}) + (1-\alpha)\mathcal{A}(\mathcal{G})$ be the convex linear combination of $\mathcal{D}(\mathcal{G})$ and $\mathcal{A}(\mathcal{G})$, where $0 \leq \alpha < 1$. The α-spectral radius of $\mathcal{G}$, denoted by $\rho_\alpha(\mathcal{G})$, is defined to be the largest modulus of all the eigenvalues of $\mathcal{A}_\alpha(\mathcal{G})$, i.e., $\rho_\alpha(\mathcal{G}) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \mathcal{A}_\alpha(\mathcal{G})\}$. Inspired by the work of Nikiforov [15], Lin et al. [12] and Guo and Zhou [6] proposed to study $\mathcal{A}_\alpha(\mathcal{G})$ and $\rho_\alpha(\mathcal{G})$. Obviously, $\rho_0(\mathcal{G})$ and $\rho_{\frac{1}{2}}(\mathcal{G})$ are respectively the spectral radius of $\mathcal{G}$ and the signless Laplacian spectral radius of $\mathcal{G}$.

Let $\boldsymbol{x}$ be a vector of dimension n and U a subset in $[n]$. We write $x^U = \prod_{i \in U} x_i$ for short. For a k-uniform hypergraph $\mathcal{G}$, by the definition of $\mathcal{A}_\alpha(\mathcal{G})$, (1) and (2), we get

$$(\mathcal{A}_\alpha(\mathcal{G})\boldsymbol{x})_v = \alpha d_v x_v^{k-1} + (1-\alpha) \sum_{e: v \in e} x^{e \setminus \{v\}}, \text{ for each } v \in V(\mathcal{G}), \tag{3}$$

$$\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}_\alpha(\mathcal{G})\boldsymbol{x}) = \alpha \sum_{v \in V(\mathcal{G})} d_v x_v^k + (1-\alpha) \sum_{e \in E(\mathcal{G})} k x^e. \tag{4}$$

Since the studies on the α-spectral radius of hypergraphs are of practical significance, they have attracted many attentions from researchers. The hypergraphs with the extremal α-spectral radii have been obtained. Among the k-uniform supertrees, You et al. [27] obtained the supertrees with the first to the third largest α-spectral radii, and they proposed a conjecture on the supertrees with the fourth to the eighth largest α-spectral radii. Wang et al. [24] solved this conjecture and the supertrees with the fourth to the eighth largest α-spectral radii among the k-uniform supertrees were obtained. Among the k-uniform non-caterpillar hypergraphs with a given diameter, Wang et al. [22] deduced the supertrees with the first and the second largest α-spectral radii. Among hypergraphs with a given number of pendent edges and among the unicyclic hypergraphs, Lin and Zhou [13] obtained the hypergraphs with the largest α-spectral radii. Among the k-uniform unicyclic hypergraphs with a fixed diameter and among the k-uniform unicyclic hypergraphs with a given number of pendent edges, Kang et al. [8] characterized the hypergraphs with the largest α-spectral radii. For the upper bounds of the α-spectral radius for hypergraphs, one can refer to Refs. [3, 6, 7, 12, 13].

In studying the spectral radius of the k-uniform hypergraphs, one of the powerful methods is the α-normal labeling method, which was first developed by Lu and Man [14]. For example, Ouyang et al. [16] used it to determine the first five hypergraphs with larger spectral radii among the k-uniform unicyclic hypergraphs and the first three hypergraphs with larger spectral radii among the k-uniform bicyclic hypergraphs. Researchers also extended the α-normal labeling method to study the upper bound of the α-spectral radius of hypergraphs [23] and the p-spectral radius of hypergraphs [10]. For more details about the α-normal labeling method, one can refer to Refs. [1, 16, 20].

Let $\mathcal{B}(n,k)$ be the set of the connected k-uniform bicyclic hypergraphs, where $k \geq 3$. Motivated by the above-mentioned results, in this article, we will study the hypergraphs with the larger α-spectral radii among $\mathcal{B}(n,k)$, where $k \geq 3$.

This article is organized as follows. In Section 2, we introduce some necessary lemmas which are useful for subsequent proofs. In Section 3, we propose a useful and new ρ_α-normal labeling method for studying the α-spectral radius of k-uniform hypergraphs. In Section 4, by using the ρ_α-normal labeling method proposed in Section 3, we compare the α-spectral radii of some hypergraphs among $\mathcal{B}(n,k)$. With the aid of some transformations and the results obtained in Section 4, we obtain the k-uniform hypergraphs with the first and the second largest α-spectral radii among $\mathcal{B}(n,k)$ in Section 5, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

2. Preliminaries

In this section, some definitions and necessary lemmas are introduced.

Definition 2.1. *[26] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$ be a nonnegative tensor of order k and dimension n. For any nonempty proper index subset $I \subset [n]$, if there is at least an entry $a_{i_1 i_2 \cdots i_k} > 0$, where $i_1 \in I$ and at least an $i_j \in [n] \setminus I$ for $j = 2, 3, \ldots, k$, then $\mathcal{A}$ is called a nonnegative weakly irreducible tensor.*

Let $\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \mid x_i \geqslant 0, \forall i \in [n]\}$ and $\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \mid x_i > 0, \forall i \in [n]\}$.

Lemma 2.2. *[5, 25] (The Perron–Frobenius theorem for nonnegative tensors) Let $\mathcal{A}$ be a nonnegative tensor of order k and dimension n, where $k \geq 2$. Then we have the following statements.*

(i). $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $\boldsymbol{x} \in \mathbb{R}^n_+$ corresponding to it.

(ii). If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with the positive eigenvector $\boldsymbol{x} \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

Lemma 2.3. *[17] A k-uniform hypergraph $\mathcal{G}$ is connected if and only if $\mathcal{A}_\alpha(\mathcal{G})$ is weakly irreducible.*

From Lemmas 2.2 and 2.3, if $\mathcal{G}$ is a connected k-uniform hypergraph, then there exists the unique vector $\boldsymbol{x} \in \mathbb{R}^n_{++}$ corresponding to $\rho_\alpha(\mathcal{G})$. This vector $\boldsymbol{x}$ is referred to as the α-Perron vector of $\mathcal{G}$, where $\|\boldsymbol{x}\|_k^k = 1$.

Lemma 2.4. *[19] Let $\mathcal{A}$ be a nonnegative symmetric tensor of order k and dimension n. We have*

$$\rho(\mathcal{A}) = \max\left\{\boldsymbol{x}^T(\mathcal{A}\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n_+, \|\boldsymbol{x}\|_k^k = 1\right\}.$$

Furthermore, $\boldsymbol{x} \in \mathbb{R}^n_+$ with $\|\boldsymbol{x}\|_k^k = 1$ is an optimal solution of the above optimization problem if and only if it is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\rho(\mathcal{A})$.

From Lemma 2.4, $\rho(\mathcal{A}_\alpha)$ can be expressed as follows:

$$\rho(\mathcal{A}_\alpha) = \max\left\{\frac{\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}_\alpha \boldsymbol{x})}{\|\boldsymbol{x}\|_k^k}, \boldsymbol{x} \in \mathbb{R}^n_+, \boldsymbol{x} \neq 0\right\}. \tag{5}$$

The edge-removing operation, which is a useful method for studying the α-spectral radius, is shown in Definition 2.5.

Definition 2.5. *[9] Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a hypergraph with $v \in V(\mathcal{G})$ and $e_1, \ldots, e_r \in E(\mathcal{G})$ such that $v \notin e_i$ for $i \in [r] = \{1, 2, \cdots, r\}$, where $r \geq 1$. Suppose that $u_i \in e_i$, where $i \in [r]$ and the vertices $u_1, u_2, \cdots u_r$ are not necessarily distinct. Let $e_i' = (e_i \backslash \{u_i\}) \cup \{v\}$, where $i \in [r]$. Let $\mathcal{G}' = (V(\mathcal{G}'), E(\mathcal{G}'))$ be the hypergraph with $E(\mathcal{G}') = (E(\mathcal{G}) \backslash \{e_1, \ldots, e_r\}) \cup \{e_1', \ldots, e_r'\}$. Then we say that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by removing the edges $(e_1, \ldots, e_r)$ from $(u_1, \ldots, u_r)$ to v.*

Lemma 2.6. *[6] Let $\mathcal{G}$ be a connected k-uniform hypergraph, and $\mathcal{G}'$ the hypergraph obtained from $\mathcal{G}$ by removing edges $(e_1, \ldots, e_r)$ from $(u_1, \ldots, u_r)$ to v, where $r \geq 1$. Let $\boldsymbol{x}$ be the α-Perron vector of $\mathcal{G}$. If $x_v \geq max\{x_{u_1}, \ldots, x_{u_r}\}$, then $\rho_\alpha(\mathcal{G}') > \rho_\alpha(\mathcal{G})$.*

Lemma 2.7. *[16] Let $\mathcal{G}$ be a simple connected r-cyclic k-uniform hypergraph with n vertices. Let $\mathcal{G}'$ be a connected subhypergraph of $\mathcal{G}$. If $\mathcal{G}'$ is r'-cyclic, then we have $r' \leq r$.*

3. A new ρ_α-normal labeling method for the α-spectral radius of k-uniform hypergraphs

In this section, we will propose a useful ρ_α-normal labeling method for the α-spectral radius of the k-uniform hypergraphs, which generalizes the α-normal labeling method developed by Lu and Man [14] for the spectral radius of the k-uniform hypergraphs. The definitions of ρ_α-normal, ρ_α-subnormal and ρ_α-supernormal for the α-spectral radius of the k-uniform hypergraphs are introduced, which are shown in Definitions 3.1–3.6, respectively. Then, we give the relationship between the ρ_α-normal labeling and the α-spectral radius of k-uniform hypergraphs, which are shown in Lemmas 3.3–3.7.

Definition 3.1. *Let $k \geq 2$ and $0 \leq \alpha < 1$. A connected k-uniform hypergraph $\mathcal{G}$ is called ρ_α-normal if there exists a weighted incidence matrix $\boldsymbol{B}$ satisfying*

(i). $\sum_{e:v\in e}\Big(B(v,e)+\alpha\Big) = \rho_\alpha$, for any $v \in V(\mathcal{G})$.

(ii). $\prod_{v:v\in e} B(v,e) = (1-\alpha)^k$, for any $e \in E(\mathcal{G})$.

Moreover, the incidence matrix $\boldsymbol{B}$ is called consistent if for any cycle $v_0e_1v_1\ldots v_l$ $(v_0 = v_l)$ of $\mathcal{G}$, we have $\prod_{i=1}^{l}\frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1$. In this case, we call $\mathcal{G}$ consistently ρ_α-normal.

Remark 3.2. *For any supertree $\mathcal{T}$, since $\mathcal{T}$ does not contain cycles, $\mathcal{T}$ satisfies the consistent condition naturally.*

Lemma 3.3. *Let $\mathcal{G}$ be a connected k-uniform hypergraph, where $k \geq 2$. The α-spectral radius of $\mathcal{G}$ is ρ_α if and only if $\mathcal{G}$ is consistently ρ_α-normal, where $0 \leq \alpha < 1$.*

Proof. Let $V(\mathcal{G}) = \{v_1, v_2, \cdots, v_n\}$.

(1). The proof of necessity.

We suppose that the α-spectral radius of $\mathcal{G}$ is ρ_α. We will prove that $\mathcal{G}$ is consistently ρ_α-normal. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ be the α-Perror eigenvector of the α-spectral radius of $\mathcal{G}$. We define the weighted incidence matrix $\boldsymbol{B}$ as follows. Let

$$B(v,e) = \begin{cases} \frac{(1-\alpha)x^e}{x_v^k}, & \text{if } v \in e, \\ 0, & \text{otherwise.} \end{cases}$$

(1.1). For any $v \in V(\mathcal{G})$, we have

$$\sum_{e:v\in e}\Big(B(v,e)+\alpha\Big) = \sum_{e:v\in e}\left(\frac{(1-\alpha)x^e}{x_v^k}+\alpha\right) = \frac{\alpha d_v x_v^k + (1-\alpha)\sum_{e:v\in e}x^e}{x_v^k}. \tag{6}$$

By the eigenequation (3) of $\mathcal{G}$ at v, we get

$$\rho_\alpha x_v^k = \alpha d_v x_v^k + (1-\alpha)\sum_{e:v\in e}x^e. \tag{7}$$

Therefore, by substituting (7) into (6), we get $\sum_{e:v\in e}\Big(B(v,e)+\alpha\Big) = \rho_\alpha$. Namely, we have Definition 3.1 (i).

(1.2). For any $e \in E(\mathcal{G})$, we get

$$\prod_{v:v\in e}B(v,e) = \prod_{v:v\in e}\frac{(1-\alpha)x^e}{x_v^k} = (1-\alpha)^k\cdot\frac{(x^e)^k}{\prod_{v:v\in e}x_v^k} = (1-\alpha)^k, \tag{8}$$

where the third equality in (8) holds since $\prod_{v:v\in e} x_v^k = (x^e)^k$. By (8), we have Definition 3.1 (ii).

Next, we prove that $\boldsymbol{B}$ is consistent. For any cycle $v_0e_1v_1\ldots v_l$ ($v_l = v_0$) of $\mathcal{G}$, we obtain

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = \prod_{i=1}^{l} \frac{\frac{(1-\alpha)x^{e_i}}{x_{v_i}^k}}{\frac{(1-\alpha)x^{e_i}}{x_{v_{i-1}}^k}} = \prod_{i=1}^{l} \frac{x_{v_{i-1}}^k}{x_{v_i}^k} = \frac{x_{v_0}^k}{x_{v_l}^k} = 1. \tag{9}$$

By (9), we get that $\mathcal{G}$ is consistently ρ_α-normal.

(2). The proof of sufficiency.

Suppose that $\mathcal{G}$ is consistently ρ_α-normal. We will prove that the α-spectral radius of $\mathcal{G}$ is ρ_α. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an arbitrary nonzero vector in $\mathbb{R}_+^n$.

For any $e \in E(\mathcal{G})$, if $\prod_{v:v\in e} B(v, e) = (1-\alpha)^k$, then we have

$$(1-\alpha) \sum_{e\in E(\mathcal{G})} \frac{k}{1-\alpha} \prod_{v:v\in e} ((B(v, e))^{\frac{1}{k}} x_v) = (1-\alpha) \sum_{e\in E(\mathcal{G})} kx^e. \tag{10}$$

By the Arithmetic Mean–Geometry Mean inequality, we get

$$\sum_{e\in E(\mathcal{G})} k \prod_{v:v\in e} (B(v, e)^{\frac{1}{k}} x_v) \le \frac{\sum_{e\in E(\mathcal{G})} \sum_{v:v\in e} kB(v, e)x_v^k}{k}. \tag{11}$$

Obviously, we have

$$\alpha \sum_{v\in V(\mathcal{G})} d_v x_v^k = \sum_{v\in V(\mathcal{G})} \sum_{e:v\in e} \alpha x_v^k. \tag{12}$$

By (4), (10)–(12) and Condition (i) in Definition 3.1, we have

$$\begin{aligned} \boldsymbol{x}^T(\mathcal{A}_\alpha(\mathcal{G})\boldsymbol{x}) &= \alpha \sum_{v\in V(\mathcal{G})} d_v x_v^k + (1-\alpha) \sum_{e\in E(\mathcal{G})} kx^e \\ &\le \sum_{v\in V(\mathcal{G})} \sum_{e:v\in e} \Big(\alpha + B(v, e)\Big) x_v^k = \rho_\alpha \sum_{v\in V(\mathcal{G})} x_v^k = \rho_\alpha \parallel \boldsymbol{x} \parallel_k^k . \end{aligned} \tag{13}$$

Therefore, by (13) and the arbitrariness of $\boldsymbol{x}$, we obtain $\rho_\alpha(\mathcal{G}) \le \rho_\alpha$, with the equality if and only if $\mathcal{G}$ is ρ_α-normal and the equality in (11) holds. Namely, there is a nonzero solution $\{x_i\}$ for the system of the following homogeneous linear equations:

$$B(v_{i_1}, e)^{\frac{1}{k}} x_{v_{i_1}} = B(v_{i_2}, e)^{\frac{1}{k}} x_{v_{i_2}} = \ldots = B(v_{i_k}, e)^{\frac{1}{k}} x_{v_{i_k}}, \forall e = \{v_{i_1}, \ldots, v_{i_k}\} \in E(\mathcal{G}). \tag{14}$$

Let v_0 be an arbitrary vertex in $V(\mathcal{G})$. For any $u \in V(\mathcal{G})$, since $\mathcal{G}$ is connected, there exists a path $v_0e_1v_1e_2v_2\ldots v_l$ ($v_l = u$) connecting v_0 and u. Let $x_{v_0}^* = 1$. For $u \in V(\mathcal{G})$, we define $x_u^* = \left(\prod_{i=1}^{l} \frac{B(v_{i-1}, e_i)}{B(v_i, e_i)}\right)^{\frac{1}{k}}$. The consistent condition guarantees that x_u^* is independent of the choice of the path. We can check that $(x_1^*, x_2^*, \cdots, x_n^*)$ is a solution of (14). Thus, we have $\rho_\alpha(\mathcal{G}) = \rho_\alpha$. □

Definition 3.4. *Let $k \ge 2$ and $0 \le \alpha < 1$. A connected k-uniform hypergraph $\mathcal{G}$ is called ρ_α-subnormal if there exists a weighted incidence matrix $\boldsymbol{B}$ satisfying*

(*i*). $\sum_{e:v\in e} \Big(B(v, e) + \alpha\Big) \le \rho_\alpha$, *for any* $v \in V(\mathcal{G})$.

(*ii*). $\prod_{v:v\in e} B(v, e) \ge (1-\alpha)^k$, *for any* $e \in E(\mathcal{G})$.

Moreover, $\mathcal{G}$ is called strictly ρ_α-subnormal if it is ρ_α-subnormal but not ρ_α-normal.

Lemma 3.5. *Let $\mathcal{G}$ be a connected k-uniform hypergraph, where $k \geq 2$. If $\mathcal{G}$ is ρ_α-subnormal, then $\rho_\alpha(\mathcal{G}) \leq \rho_\alpha$, where $0 \leq \alpha < 1$. Moreover, if $\mathcal{G}$ is strictly ρ_α-subnormal, then $\rho_\alpha(\mathcal{G}) < \rho_\alpha$.*

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an arbitrary nonzero vector in $\mathbb{R}^n_+$. For any $e \in E(\mathcal{G})$, if $\prod\limits_{v:v\in e} B(v, e) \geq (1-\alpha)^k$, then we have

$$(1-\alpha)\sum_{e\in E(\mathcal{G})}\frac{k}{1-\alpha}\prod_{v:v\in e}\Big((B(v,e))^{\frac{1}{k}}\,x_v\Big) \geq (1-\alpha)\sum_{e\in E(\mathcal{G})} kx^e. \tag{15}$$

By (4), (11), (12), (15), and Condition (i) in Definition 3.4, we have

$$\begin{aligned}\boldsymbol{x}^T(\mathcal{A}_\alpha(\mathcal{G})\boldsymbol{x}) &= \alpha\sum_{v\in V(\mathcal{G})} d_v x_v^k + (1-\alpha)\sum_{e\in E(\mathcal{G})} kx^e \\ &\leq \sum_{v\in V(\mathcal{G})}\sum_{e:v\in e}\Big(\alpha + B(v,e)\Big)x_v^k \leq \rho_\alpha \sum_{v\in V(\mathcal{G})} x_v^k = \rho_\alpha \parallel \boldsymbol{x} \parallel_k^k .\end{aligned} \tag{16}$$

Therefore, by (16) and the arbitrariness of $\boldsymbol{x}$, we obtain $\rho_\alpha(\mathcal{G}) \leq \rho_\alpha$. If $\mathcal{G}$ is strictly ρ_α-subnormal, then the inequality in (15) or the second inequality in (16) holds. Thus, we get $\rho_\alpha(\mathcal{G}) < \rho_\alpha$. □

Definition 3.6. *Let $k \geq 2$ and $0 \leq \alpha < 1$. A connected k-uniform hypergraph $\mathcal{G}$ is called ρ_α-supernormal if there exists a weighted incidence matrix $\boldsymbol{B}$ satisfying*

(i). $\sum\limits_{e:v\in e}\Big(B(v,e)+\alpha\Big) \geq \rho_\alpha$, *for any $v \in V(\mathcal{G})$.*

(ii). $\prod\limits_{v:v\in e} B(v,e) \leq (1-\alpha)^k$, *for any $e \in E(\mathcal{G})$.*

Moreover, $\mathcal{G}$ is called strictly ρ_α-supernormal if it is ρ_α-supernormal but not ρ_α-normal.

Lemma 3.7. *Let $\mathcal{G}$ be a connected k-uniform hypergraph, where $k \geq 2$. If G is consistently ρ_α-supernormal, then $\rho_\alpha(\mathcal{G}) \geq \rho_\alpha$, where $0 \leq \alpha < 1$. Moreover, if $\mathcal{G}$ is strictly consistently ρ_α-supernormal, then $\rho_\alpha(\mathcal{G}) > \rho_\alpha$.*

Proof. From the consistent condition of $\mathcal{G}$ and the proof of sufficiency of Lemma 3.3, there exists an eigenvector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n_+$ such that (14) holds. We have

$$\sum_{e\in E(\mathcal{G})}\prod_{v:v\in e}\Big(B(v,e)^{\frac{1}{k}}x_v\Big) = \sum_{e\in E(\mathcal{G})}\frac{\sum_{v:v\in e}B(v,e)x_v^k}{k}. \tag{17}$$

For any $e \in E(\mathcal{G})$, if $\prod\limits_{v:v\in e} B(v, e) \leq (1-\alpha)^k$, then we obtain

$$(1-\alpha)\sum_{e\in E(\mathcal{G})}\frac{k}{1-\alpha}\prod_{v:v\in e}((B(v,e))^{\frac{1}{k}}x_v) \leq (1-\alpha)\sum_{e\in E(\mathcal{G})} kx^e. \tag{18}$$

By (4), (12), (17), (18), and Condition (i) in Definition 3.6, we get

$$\begin{aligned}\boldsymbol{x}^T(\mathcal{A}_\alpha(\mathcal{G})\boldsymbol{x}) &= \alpha\sum_{v\in V(\mathcal{G})} d_v x_v^k + (1-\alpha)\sum_{e\in E(\mathcal{G})} kx^e \\ &\geq \sum_{v\in V(\mathcal{G})}\sum_{e:v\in e}\Big(\alpha + B(v,e)\Big)x_v^k \geq \rho_\alpha \sum_{v\in V(\mathcal{G})} x_v^k = \rho_\alpha \parallel \boldsymbol{x} \parallel_k^k .\end{aligned} \tag{19}$$

Therefore, by (19), we obtain $\rho_\alpha(\mathcal{G}) \geq \frac{\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}_\alpha \boldsymbol{x})}{\|\boldsymbol{x}\|_k^k} \geq \rho_\alpha$. If $\mathcal{G}$ is strictly consistently ρ_α-supernormal, then the inequality in (18) or the second inequality in (19) holds. Thus, we get $\rho_\alpha(\mathcal{G}) > \rho_\alpha$. □

4. Comparing the α-spectral radii of some hypergraphs among $\mathcal{B}(n,k)$

In this section, we will use the ρ_α-normal labeling method proposed in Section 3 to compare the α-spectral radii of some hypergraphs among $\mathcal{B}(n,k)$.

Some definitions of hypergraphs in $\mathcal{B}(n,k)$ are introduced firstly. Let e_1, e_2, e_3, and e_4 be four edges with k vertices, where $k \geq 3$. Let $e_1 = \{u_1, u_2, u_3, w_{i_1}, \dots w_{i_{k-3}}\}$, $e_2 = \{u_1, u_2, u_3, w'_{i_1}, \dots w'_{i_{k-3}}\}$, $e_3 = \{u_1, u_2, u_3, w''_{i_1}, \dots w''_{i_{k-3}}\}$, and $e_4 = \{u_1, u_2, u_3, w'''_{i_1}, \dots w'''_{i_{k-3}}\}$. Let $\mathcal{A}$, $\mathcal{B}$, C, $\mathcal{D}$, and $\mathcal{F}$ be the five hypergraphs as shown in Fig. 2. In $\mathcal{A}$, u_1 and u_2 are simultaneously incident with e_1, e_2 and e_3 and $d_{\mathcal{A}}(v) = 1$ for $v \in V(\mathcal{A})\backslash\{u_1, u_2\}$. In $\mathcal{B}$, $d_{\mathcal{B}}(u_1) = d_{\mathcal{B}}(u_2) = d_{\mathcal{B}}(u_3) = 2$ and $d_{\mathcal{B}}(v) = 1$ for $v \in V(\mathcal{B})\backslash\{u_1, u_2, u_3\}$. In C, $d_C(u_1) = 4$, $d_C(u_2) = d_C(u_3) = 2$ and $d_C(v) = 1$ for $v \in V(C)\backslash\{u_1, u_2, u_3\}$. $\mathcal{D}$ is obtained from e_1, e_2, e_3, and e_4 by identifying u_1 of e_1, e_3, and e_4 together, identifying u_2 of e_2, e_3, and e_4 together, and identifying u_3 of e_1 and e_2 together. $\mathcal{F}$ is obtained from e_1, e_2, and e_3 by identifying u_1 of e_1, e_2, and e_3 together, identifying u_2 of e_1 and e_2 together, and identifying u_3 of e_1 and e_3 together.

A hyperstar with a edges, denoted by $\mathcal{S}_a$ ($a \geq 1$), is a k-uniform supertee such that it has only one vertex (denoted by u_0) of degree a and all the other vertices have degree 1. Namely, in $\mathcal{S}_a$, all the edges of $\mathcal{S}_a$ are incident with the common vertex u_0. We refer to u_0 as the center vertex of $\mathcal{S}_a$. For a hypergraph $\mathcal{H}$ and $v \in V(\mathcal{H})$, if we identify v of $\mathcal{H}$ with u_0 of a hyperstar S_a, then we say that the resulting hypergraph is obtained from $\mathcal{H}$ by attaching S_a at v.

Let $m = \frac{n+1}{k-1} \geq 5$. We assume that a, b and c are nonnegative integers.

Let $\mathcal{A}_{n,k}(a,b)$ be the hypergraph obtained from $\mathcal{A}$ by attaching hyperstars $\mathcal{S}_a$ and $\mathcal{S}_b$ at u_1 and u_2 of $\mathcal{A}$, respectively, where $a \geq b \geq 0$ and $a+b+c = m-3$. $\mathcal{A}_{n,k}(a,b)$ is shown in Fig. 1. Let $\mathcal{A}'_{n,k}(a,b,c)$ be the hypergraph obtained from $\mathcal{A}_{n,k}(a,b)$ by attaching a hyperstar $\mathcal{S}_c$ at a core vertex (denoted by u_3) in e_1, where $a \geq b \geq 0$, $c > 1$ and $a+b+c = m-3$. Let $\mathcal{A}^*_{n,k}(a,b,c)$ be the hypergraph obtained from $\mathcal{A}_{n,k}(a+1,b)$ by attaching a hyperstar $\mathcal{S}_c$ at a core vertex (denoted by u_3) in an edge of $\mathcal{S}_a$, where $a, b \geq 0$, $c > 1$ and $a+b+c = m-4$. Let $\mathcal{B}_{n,k}(a,b,c)$, $C_{n,k}(a,b,c)$, $\mathcal{D}_{n,k}(a,b,c)$, and $\mathcal{F}_{n,k}(a,b,c)$ be the hypergraphs obtained respectively from $\mathcal{B}$, C, $\mathcal{D}$, and $\mathcal{F}$ by attaching hyperstars $\mathcal{S}_a$, $\mathcal{S}_b$ and $\mathcal{S}_c$ at u_1, u_2 and u_3. It is noted that $a+b+c = m-2$ for $\mathcal{B}_{n,k}(a,b,c)$, $a+b+c = m-4$ for $C_{n,k}(a,b,c)$, and $a+b+c = m-3$ for $\mathcal{D}_{n,k}(a,b,c)$ and $\mathcal{F}_{n,k}(a,b,c)$. For example, $\mathcal{A}'_{n,k}(a,b,c)$, $A^*_{n,k}(a,b,c)$, $\mathcal{B}_{n,k}(a,b,c)$, $C_{n,k}(a,b,c)$, $\mathcal{D}_{n,k}(a,b,c)$, and $\mathcal{F}_{n,k}(a,b,c)$ are shown in Fig. 3.

For simplicity, let $\mathcal{A}_{n,k}(m-3,0) = \mathcal{A}^{(1)}_{n,k}$ with $m \geq 4$, $\mathcal{A}_{n,k}(m-4,1) = \mathcal{A}^{(2)}_{n,k}$ with $m \geq 5$, $A'_{n,k}(0,0,m-3) = \mathcal{A}^{(3)}_{n,k}$ with $m \geq 4$, $\mathcal{B}_{n,k}(m-2,0,0) = \mathcal{B}^{(1)}_{n,k}$ with $m \geq 3$, $\mathcal{B}_{n,k}(m-3,1,0) = \mathcal{B}^{(2)}_{n,k}$ with $m \geq 4$, $C_{n,k}(m-4,0,0) = C_{n,k}$ with $m \geq 5$, and $\mathcal{F}_{n,k}(m-3,0,0) = \mathcal{F}_{n,k}$ with $m \geq 4$. Let $\mathcal{B}^{(3)}_{n,k}$ be the hypergraph obtained from $\mathcal{B}$ by attaching a hyperstar $\mathcal{S}_{m-2}$ at a core vertex (denoted by u_4) in e_1, where $m \geq 3$. $\mathcal{B}^{(3)}_{n,k}$ is shown in Fig. 4.

For a hypergraph $\mathcal{H} \in \mathcal{B}(n,k)$, if we repeatedly delete the pendent edges of $\mathcal{H}$, then we get a resulting hypergraph such that it has no pendent edges. We denote the resulting hypergraph by $\widehat{\mathcal{H}}$ and call $\widehat{\mathcal{H}}$ the base hypergraph of $\mathcal{H}$. Since $\mathcal{H}$ is a connected 2-cyclic hypergraph, the number of IVs in $\widehat{\mathcal{H}}$ is at least two. According to the numbers of the IVs in $\mathcal{H}$, we have $\mathcal{B}(n,k) = \bigcup_{i=2}^{n} \mathcal{B}_i(n,k)$, where $\mathcal{B}_i(n,k)$ is the subset of $\mathcal{B}(n,k)$ in which each hypergraph has exactly i IVs, where $i \geq 2$. Obviously, if $i = 2$, since $\mathcal{H}$ is a bicyclic hypergraph, the two IVs of $\mathcal{H}$ must be incident with three common edges, namely $\widehat{\mathcal{H}} = \mathcal{A}$. Furthermore, if $\mathcal{H} \in \mathcal{B}_2(n,k)$, when $m = \frac{n+1}{k-1} \geq 4$, we get $\mathcal{H} = \mathcal{A}_{n,k}(a,b)$. If $\mathcal{H} \in \mathcal{B}_3(n,k)$, since $\mathcal{H}$ is a bicyclic hypergraph, bearing Lemma 2.7 in mind, we get $\widehat{\mathcal{H}} = \{\mathcal{A}, \mathcal{B}, C, \mathcal{D}, \mathcal{F}\}$. Thus, we have

$$\mathcal{B}_3(n,k) = \{\mathcal{A}'_{n,k}(a,b,c), A^*_{n,k}(a,b,c), \mathcal{B}_{n,k}(a,b,c), C_{n,k}(a,b,c), \mathcal{D}_{n,k}(a,b,c), \mathcal{F}_{n,k}(a,b,c)\}. \tag{20}$$

Ouyang et al. [16] obtained the hypergraphs with the first, the second, and the third largest spectral radii among $\mathcal{B}(n,k)$, which are shown in Lemma 4.1.

Lemma 4.1. *[16] Let $\mathcal{H} \in \mathcal{B}(n,k) \setminus \{\mathcal{A}^{(1)}_{n,k}, \mathcal{B}^{(1)}_{n,k}, \mathcal{A}^{(2)}_{n,k}\}$, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 5$. We have $\rho\left(\mathcal{A}^{(1)}_{n,k}\right) = \rho\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho\left(\mathcal{A}^{(2)}_{n,k}\right) > \rho(\mathcal{H})$.*

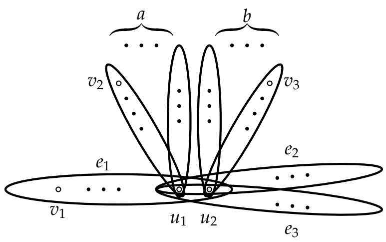
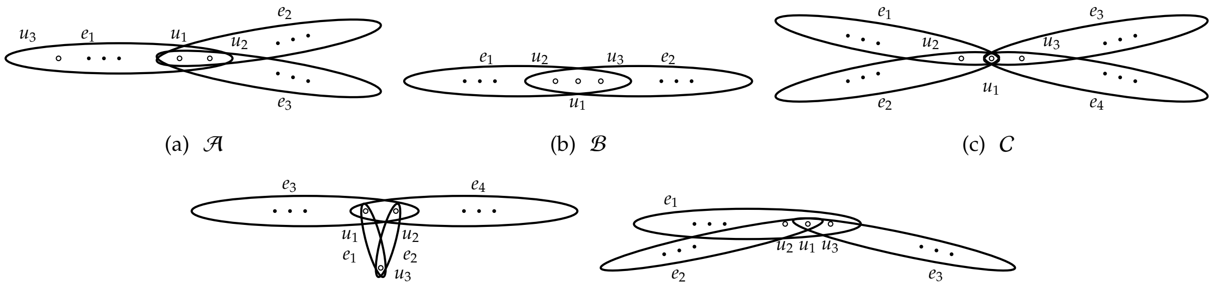


Figure 1: Bicyclic hypergraphs with two IVs: $\mathcal{A}_{n,k}(a,b)$



(a) $\mathcal{A}$ (b) $\mathcal{B}$ (c) C

(d) $\mathcal{D}$ (e) $\mathcal{F}$

Figure 2: The base hypergraphs of bicyclic hypergraphs with three IVs

To obtain the hypergraphs with the larger α-spectral radii among $\mathcal{B}(n,k)$, we introduce Lemmas 4.2–5.8 firstly.

Lemma 4.2. *Let $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. We have $\rho_\alpha\Big(\mathcal{A}^{(1)}_{n,k}\Big) \geq \rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > max\{\rho_\alpha\Big(C_{n,k}\Big), \rho_\alpha\Big(\mathcal{F}_{n,k}\Big)\}$ with the equality if and only if $\alpha = 0$.*

Proof. Let $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

In $\mathcal{B}_{n,k}(a,b,c)$ (as shown in Fig. 3(c)), let $a = m-3$ and $b = c = 0$. Namely, we get $\mathcal{B}^{(1)}_{n,k}$. Let v_1 and v_2 be the two core vertices among $\mathcal{B}^{(1)}_{n,k}$ which are respectively incident with e_1 and a pendent edge incident with u_1, where v_1 and v_2 of $\mathcal{B}^{(1)}_{n,k}$ are shown in Fig. 3(c). Let $\rho^\Delta_\alpha = \rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big)$ and $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n_{++}$ be the α-Perron vector of ρ^Δ_α. We suppose that $\mathcal{B}^{(1)}_{n,k}$ is consistently ρ^Δ_α-normal. By the eigenequations (3) of $\mathcal{B}^{(1)}_{n,k}$ at v_1, v_2, u_1, and u_2 and bearing the symmetry of the entries in $\boldsymbol{x}$ in mind, we get

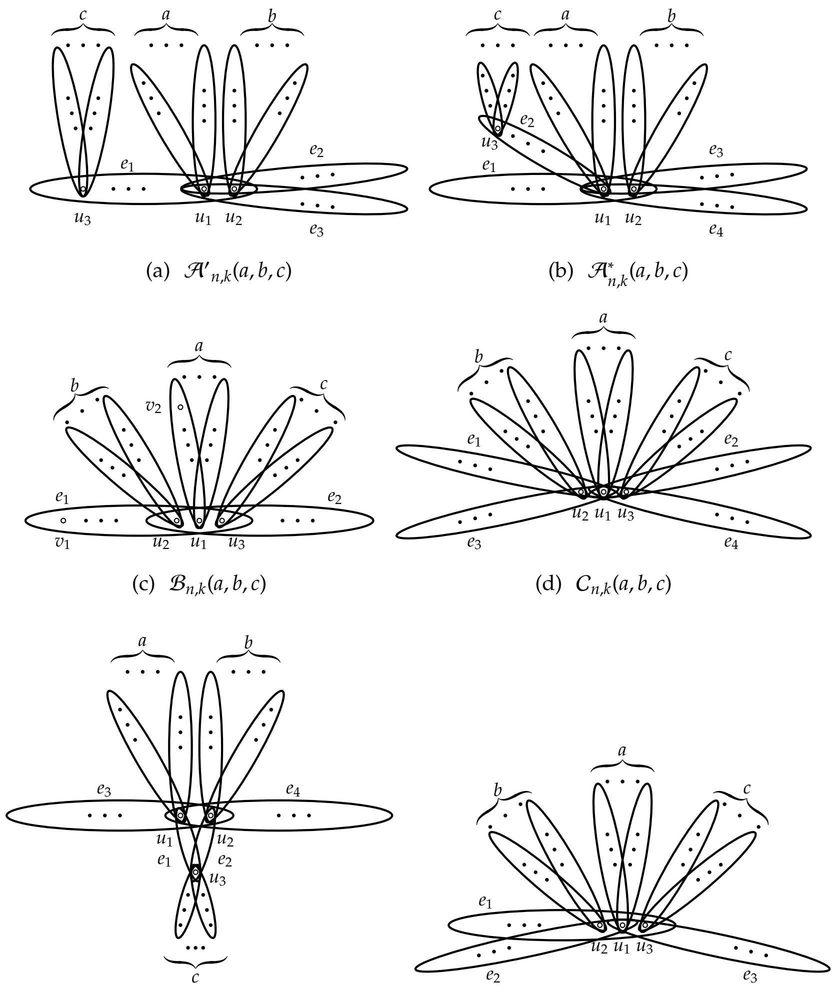
$$\rho^\Delta_\alpha x^{k-1}_{v_1} = \alpha x^{k-1}_{v_1} + (1-\alpha)x^{k-4}_{v_1}x_{u_1}x^2_{u_2}, \tag{21}$$
$$\rho^\Delta_\alpha x^{k-1}_{v_2} = \alpha x^{k-1}_{v_2} + (1-\alpha)x^{k-2}_{v_2}x_{u_1}, \tag{22}$$
$$\rho^\Delta_\alpha x^{k-1}_{u_1} = m\alpha x^{k-1}_{u_1} + (m-2)(1-\alpha)x^{k-1}_{v_2} + 2(1-\alpha)x^{k-3}_{v_1}x^2_{u_2}, \tag{23}$$
$$\rho^\Delta_\alpha x^{k-1}_{u_2} = 2\alpha x^{k-1}_{u_2} + 2(1-\alpha)x^{k-3}_{v_1}x_{u_1}x_{u_2}. \tag{24}$$

From (21), we have $\rho^\Delta_\alpha - \alpha > 0$ when $\boldsymbol{x} \in \mathbb{R}^n_{++}$ and $0 \leq \alpha < 1$. For simplicity, let

$$A_0 = \frac{1-\alpha}{\rho^\Delta_\alpha - \alpha}, \;\; A_1 = (\rho^\Delta_\alpha - \alpha)A^k_0. \tag{25}$$

Thus we have $A_0 > 0$ and $A_1 > 0$ since $\rho^\Delta_\alpha - \alpha > 0$ and $0 \leq \alpha < 1$. Furthermore, it follows from (21), (22) and (25) that

$$x_{v_1} = (A_0 x_{u_1} x^2_{u_2})^{\frac{1}{3}}, \quad x_{v_2} = A_0 x_{u_1}. \tag{26}$$



(a) $\mathcal{A}'_{n,k}(a,b,c)$ (b) $\mathcal{A}^*_{n,k}(a,b,c)$

(c) $\mathcal{B}_{n,k}(a,b,c)$ (d) $\mathcal{C}_{n,k}(a,b,c)$

(e) $\mathcal{D}_{n,k}(a,b,c)$ (f) $\mathcal{F}_{n,k}(a,b,c)$

Figure 3: Bicyclic hypergraphs with three IVs.

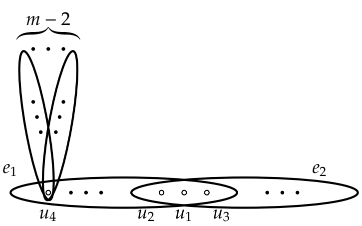


Figure 4: $\mathcal{B}^{(3)}_{n,k}$.

By combining (24) with (26), we get

$$\rho_\alpha^\Delta > 2\alpha, \quad (\text{since } 0 \le \alpha < 1 \text{ and } \boldsymbol{x} \in \mathbb{R}_{++}^n), \tag{27}$$

$$x_{u_2} = \frac{2^{\frac{3}{k}}(1-\alpha)}{(\rho_\alpha^\Delta - 2\alpha)^{\frac{3}{k}}(\rho_\alpha^\Delta - \alpha)^{1-\frac{3}{k}}} x_{u_1}. \tag{28}$$

For simplicity, let

$$B_1 = \frac{(\rho_\alpha^\Delta - \alpha)^2}{(\rho_\alpha^\Delta - 2\alpha)^2} A_1, \qquad B_2 = \frac{\rho_\alpha^\Delta - \alpha}{\rho_\alpha^\Delta - 2\alpha} A_1. \tag{29}$$

By substituting (25), (26), (28), and (29) into (23), we obtain

$$\rho_\alpha^\Delta = m\alpha + (m-2)A_1 + 8B_1. \tag{30}$$

From (29), we get $B_1 \ge B_2 > 0$ since $A_1 > 0$ and $\rho_\alpha^\Delta - \alpha \ge \rho_\alpha^\Delta - 2\alpha > 0$ (by (27)). Since $A_1, B_1 > 0$ and $0 \le \alpha < 1$, by (30), when $m \ge 20$, we obtain

$$\rho_\alpha^\Delta - 3\alpha = (m-3)\alpha + (m-2)A_1 + 8B_1 > 0. \tag{31}$$

(1.1). The proof of $\rho_\alpha\left(\mathcal{A}_{n,k}^{(1)}\right) \ge \rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right)$ for $k \ge 4$ and $m = \frac{n+1}{k-1} \ge 20$.

In $\mathcal{A}_{n,k}(a,b)$ (as shown in Fig. 1), let $a = m-3$ and $b = 0$ and we get $\mathcal{A}_{n,k}^{(1)}$. The vertices u_1, u_2, v_1, v_2, and v_3 of $\mathcal{A}_{n,k}^{(1)}$ and the edges e_1, e_2 and e_3 of $\mathcal{A}_{n,k}^{(1)}$ are shown in Fig. 1. Let $e_4, e_5, \cdots, e_m$ be the $m-3$ pendent edges of $\mathcal{A}_{n,k}^{(1)}$ attached at u_1. We construct a weighted incidence matrix $\boldsymbol{B}$ for $\mathcal{A}_{n,k}^{(1)}$ as follows. Let $B(v, e_i) = \rho_\alpha^\Delta - \alpha$, where v is an arbitrary core vertex in $V\left(\mathcal{A}_{n,k}^{(1)}\right)$ and e_i ($1 \le i \le m$) is the edge incident with v. Furthermore, let

$$B(u_1, e_1) = B(u_1, e_2) = B(u_1, e_3) = \frac{3(\rho_\alpha^\Delta - \alpha)}{\rho_\alpha^\Delta - 3\alpha} A_1,$$
$$B(u_1, e_i) = A_1, \quad \text{where } 4 \le i \le m,$$
$$B(u_2, e_1) = B(u_2, e_2) = B(u_2, e_3) = \frac{1}{3}\rho_\alpha^\Delta - \alpha.$$

When $m \ge 20$, since $A_1 > 0$ and $\rho_\alpha^\Delta - \alpha > \rho_\alpha^\Delta - 3\alpha > 0$ (by (31)), we have $B(v, e) > 0$, where v is an arbitrary vertex in $\mathcal{A}_{n,k}^{(1)}$ and e is the edge incident with v in $\mathcal{A}_{n,k}^{(1)}$. We can check $\prod\limits_{v:v\in e} B(v,e) = (1-\alpha)^k$, where $e = e_i$ ($1 \le i \le m$) is an arbitrary edge in $E\left(\mathcal{A}_{n,k}^{(1)}\right)$. For any core vertex $v \in V\left(\mathcal{A}_{n,k}^{(1)}\right)$ and $v = u_2$, we have $\sum\limits_{e:v\in e}\left(B(v,e) + \alpha\right) = \rho_\alpha^\Delta$.

Next, we compare $\sum\limits_{e:u_1\in e}\left(B(u_1,e) + \alpha\right)$ with ρ_α^Δ. Since $\rho_\alpha^\Delta - \alpha \ge \rho_\alpha^\Delta - 3\alpha > 0$ and $A_1 > 0$, we obtain $\frac{\rho_\alpha^\Delta - \alpha}{\rho_\alpha^\Delta - 3\alpha} A_1 \ge A_1$. Considering $(\rho_\alpha^\Delta - 2\alpha)^2 \ge (\rho_\alpha^\Delta - \alpha)(\rho_\alpha^\Delta - 3\alpha) > 0$ and $A_1 > 0$, we have $\frac{\rho_\alpha^\Delta - \alpha}{\rho_\alpha^\Delta - 3\alpha} A_1 \ge \frac{(\rho_\alpha^\Delta - \alpha)^2}{(\rho_\alpha^\Delta - 2\alpha)^2} A_1 = B_1$ (by (29)). Therefore, by (30), we get

$$\begin{aligned}\rho_\alpha^\Delta - \sum_{e:u_1\in e}\left(B(u_1,e) + \alpha\right) &= \rho_\alpha^\Delta - \left(3B(u_1,e_1) + (m-3)B(u_1,e_4) + m\alpha\right) \\ &= A_1 + 8B_1 - \frac{9(\rho_\alpha^\Delta - \alpha)}{\rho_\alpha^\Delta - 3\alpha} A_1 \le 0. \end{aligned} \tag{32}$$

It is noted that the third equality in (32) holds if and only if $\alpha = 0$. Therefore, if $0 < \alpha < 1$, $\mathcal{A}_{n,k}^{(1)}$ is strictly ρ_α^Δ-supernormal. Next, we verify that $\boldsymbol{B}$ is consistent. For the three cycles $u_1e_1u_2e_2u_1$, $u_1e_1u_2e_3u_1$, and

$u_1e_2u_2e_3u_1$ in $\mathcal{A}^{(1)}_{n,k}$, we have $\frac{B(u_2,e_1)}{B(u_1,e_1)}\frac{B(u_1,e_2)}{B(u_2,e_2)} = 1$, $\frac{B(u_2,e_1)}{B(u_1,e_1)}\frac{B(u_1,e_3)}{B(u_2,e_3)} = 1$, and $\frac{B(u_2,e_2)}{B(u_1,e_2)}\frac{B(u_1,e_3)}{B(u_2,e_3)} = 1$, respectively. By Lemma 3.7, we obtain $\rho_\alpha\left(\mathcal{A}^{(1)}_{n,k}\right) > \rho^\Delta_\alpha = \rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right)$ for $0 < \alpha < 1$. If $\alpha = 0$, from Lemma 4.1, we have $\rho_\alpha\left(\mathcal{A}^{(1)}_{n,k}\right) = \rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 5$.

(1.2). The proof of $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(C_{n,k}\right)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

In $C_{n,k}(a,b,c)$ (as shown in Fig. 3(d)), let $a = m-4$ and $b = c = 0$ and we get $C_{n,k}$. The vertices u_1, u_2, u_3 of $C_{n,k}$ and the edges e_1, e_2, e_3, and e_4 of $C_{n,k}$ are shown in Fig. 3(d). Let $e_5, e_6, \cdots, e_m$ be the $m-4$ pendent edges of $C_{n,k}$ attached at u_1. We construct a weighted incidence matrix $\boldsymbol{B}$ for $C_{n,k}$ as follows. Let $B(v,e_i) = \rho^\Delta_\alpha - \alpha$, where v is an arbitrary core vertex in $V\left(C_{n,k}\right)$ and e_i $(1 \leq i \leq m)$ is the edge incident with v. Furthermore, let

$$\begin{aligned}&B(u_1,e_1) = B(u_1,e_3) = B(u_1,e_2) = B(u_1,e_4) = 2B_2,\\&B(u_1,e_i) = A_1, \quad \text{where } 5 \leq i \leq m,\\&B(u_2,e_1) = B(u_2,e_2) = B(u_3,e_3) = B(u_3,e_4) = \frac{1}{2}\rho^\Delta_\alpha - \alpha.\end{aligned}$$

Since $A_1, B_2 > 0$ and $\rho^\Delta_\alpha - 2\alpha > 0$ (by (27)), we get $B(v,e) > 0$ for any vertex v and any edge e incident with v in $C_{n,k}$. We can check $\prod_{v:v\in e} B(v,e) = (1-\alpha)^k$, where $e = e_i$ $(1 \leq i \leq m)$ is an arbitrary edge in $E\left(C_{n,k}\right)$. For any core vertex $v \in V\left(C_{n,k}\right)$, $v = u_2$ and $v = u_3$, we have $\sum_{e:v\in e}\left(B(v,e)+\alpha\right) = \rho^\Delta_\alpha$.

Next, we compare $\sum_{e:u_1\in e}\left(B(u_1,e)+\alpha\right)$ with ρ^Δ_α. By (30), we get

$$\begin{aligned}\rho^\Delta_\alpha - \sum_{e:u_1\in e}\left(B(u_1,e)+\alpha\right) &= \rho^\Delta_\alpha - \left[4B(u_1,e_1) + (m-4)B(u_1,e_5) + m\alpha\right]\\&= 2A_1 + 8B_1 - 8B_2 > 0. \qquad (33)\end{aligned}$$

It is noted that (33) is deduced from $A_1 > 0$ and $B_1 \geq B_2 > 0$ (by (29)). Therefore, $C_{n,k}$ is strictly ρ^Δ_α-subnormal. By Lemma 3.5, we obtain $\rho^\Delta_\alpha = \rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(C_{n,k}\right)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

(1.3). The proof of $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

In $\mathcal{F}_{n,k}(a,b,c)$ (as shown in Fig. 3(f)), let $a = m-3$ and $b = c = 0$ and we get $\mathcal{F}_{n,k}$. The vertices u_1, u_2, and u_3 of $\mathcal{F}_{n,k}$ and the edges e_1, e_2 and e_3 of $\mathcal{F}_{n,k}$ are shown in Fig. 3(f). Let $e_4, e_5, \cdots, e_m$ be the $m-3$ pendent edges of $\mathcal{F}_{n,k}$ attached at u_1. We construct a weighted incidence matrix $\boldsymbol{B}$ for $\mathcal{F}_{n,k}$ as follows. Let $B(v,e_i) = \rho^\Delta_\alpha - \alpha$, where v is an arbitrary core vertex in $V\left(\mathcal{F}_{n,k}\right)$ and e_i $(1 \leq i \leq m)$ is the edge incident with v. Furthermore, let

$$\begin{aligned}&B(u_1,e_1) = 4B_1, \quad B(u_1,e_2) = B(u_1,e_3) = 2B_2,\\&B(u_1,e_i) = A_1, \quad \text{where } 4 \leq i \leq m,\\&B(u_2,e_1) = B(u_2,e_2) = B(u_3,e_1) = B(u_3,e_3) = \frac{1}{2}\rho^\Delta_\alpha - \alpha.\end{aligned}$$

Since $A_1, B_1, B_2 > 0$ and $\rho^\Delta_\alpha - 2\alpha > 0$ (by (27)), we can check that $B(v,e) > 0$ for any vertex v and any edge e incident with v in $\mathcal{F}_{n,k}$. It can be verified that $\prod_{v:v\in e} B(v,e) = (1-\alpha)^k$, where $e = e_i$ $(1 \leq i \leq m)$ is an arbitrary edge in $E\left(\mathcal{F}_{n,k}\right)$. For any core vertex $v \in V\left(\mathcal{F}_{n,k}\right)$, $v = u_2$ and $v = u_3$, we have $\sum_{e:v\in e}\left(B(v,e)+\alpha\right) = \rho^\Delta_\alpha$.

Next, we compare $\sum_{e:u_1\in e}\left(B(u_1,e)+\alpha\right)$ with ρ^Δ_α. By (30), we get

$$\begin{aligned}\rho^\Delta_\alpha - \sum_{e:u_1\in e}\left(B(u_1,e)+\alpha\right) &= \rho^\Delta_\alpha - \left(B(u_1,e_1) + 2B(u_1,e_2) + (m-3)B(u_1,e_4) + m\alpha\right)\\&= A_1 + 4B_1 - 4B_2 > 0. \qquad (34)\end{aligned}$$

It is noted that (34) follows from $B_1 \geq B_2 > 0$ (by (29)) and $A_1 > 0$. By (34), we obtain that $\mathcal{F}_{n,k}$ is strictly ρ^Δ_α-subnormal. By Lemma 3.5, we get $\rho^\Delta_\alpha = \rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. □

Lemma 4.3. *Let $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. We have* $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right) > max\{\rho_\alpha\left(\mathcal{B}^{(2)}_{n,k}\right), \rho_\alpha\left(\mathcal{A}^{(3)}_{n,k}\right), \rho_\alpha\left(\mathcal{B}^{(3)}_{n,k}\right)\}$.

Proof. In $\mathcal{A}_{n,k}(a, b)$ (as shown in Fig. 1), let $a = m-4$ and $b = 1$ and we get $\mathcal{A}^{(2)}_{n,k}$. The vertices u_1, u_2, v_1, v_2, and v_3 of $\mathcal{A}^{(2)}_{n,k}$ and the edges e_1, e_2 and e_3 of $\mathcal{A}^{(2)}_{n,k}$ are shown in Fig. 1, where v_1 , v_2 and v_3 are three core vertices which are respectively incident with e_1, a pendent edge incident with u_1 and the pendent edge incident with u_2 of $\mathcal{A}^{(2)}_{n,k}$. Let $e_4, e_5, \cdots, e_m$ be the $m-3$ pendent edges of $\mathcal{A}^{(2)}_{n,k}$ attached at u_1. Let $\rho^\circ_\alpha = \rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right)$ and $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n_{++}$ be the α-Perron vector of ρ°_α. We suppose that $\mathcal{A}^{(2)}_{n,k}$ is consistently ρ°_α-normal. By the eigenequations (3) of $\mathcal{A}^{(2)}_{n,k}$ at v_1, v_2, v_3 , u_1, and u_2 and bearing the symmetry of the entries in x in mind, we get

$$\rho^\circ_\alpha x^{k-1}_{v_1} = \alpha x^{k-1}_{v_1} + (1-\alpha)x^{k-3}_{v_1} x_{u_1} x_{u_2}, \tag{35}$$
$$\rho^\circ_\alpha x^{k-1}_{v_2} = \alpha x^{k-1}_{v_2} + (1-\alpha)x^{k-2}_{v_2} x_{u_1}, \tag{36}$$
$$\rho^\circ_\alpha x^{k-1}_{v_3} = \alpha x^{k-1}_{v_3} + (1-\alpha)x^{k-2}_{v_3} x_{u_2}, \tag{37}$$
$$\rho^\circ_\alpha x^{k-1}_{u_1} = (m-1)\alpha x^{k-1}_{u_1} + (m-4)(1-\alpha)x^{k-1}_{v_2} + 3(1-\alpha)x^{k-2}_{v_1} x_{u_2}, \tag{38}$$
$$\rho^\circ_\alpha x^{k-1}_{u_2} = 4\alpha x^{k-1}_{u_2} + (1-\alpha)x^{k-1}_{v_3} + 3(1-\alpha)x^{k-2}_{v_1} x_{u_1}. \tag{39}$$

From (35), we have $\rho^\circ_\alpha - \alpha > 0$ when $x \in \mathbb{R}^n_{++}$ and $0 \leq \alpha < 1$. Furthermore, it follows from (35)–(37) that, respectively,

$$x_{v_1} = \sqrt{\frac{1-\alpha}{\rho^\circ_\alpha - \alpha} x_{u_1} x_{u_2}}, \quad x_{v_2} = \frac{1-\alpha}{\rho^\circ_\alpha - \alpha} x_{u_1}, \quad x_{v_3} = \frac{1-\alpha}{\rho^\circ_\alpha - \alpha} x_{u_2}. \tag{40}$$

By combining (39) with (40), we get

$$\rho^\Delta_\alpha - 4\alpha - \frac{(1-\alpha)^k}{(\rho^\circ_\alpha - \alpha)^{k-1}} > 0, \ \text{(since } 0 \leq \alpha < 1, \ \rho^\circ - \alpha > 0 \text{ and } x \in \mathbb{R}^n_{++}), \tag{41}$$

$$x_{u_2} = \frac{3^{\frac{2}{k}}(1-\alpha)}{(\rho^\circ_\alpha - 4\alpha - \frac{(1-\alpha)^k}{(\rho^\circ_\alpha - \alpha)^{k-1}})^{\frac{2}{k}} (\rho^\circ_\alpha - \alpha)^{1-\frac{2}{k}}} x_{u_1}. \tag{42}$$

For simplicity, let

$$A_2 = \frac{(1-\alpha)^k}{(\rho^\circ_\alpha - \alpha)^{k-1}}, \quad C_1 = \frac{\rho^\circ_\alpha - \alpha}{\rho^\circ_\alpha - 4\alpha - A_2} A_2, \quad C_2 = \frac{(\rho^\circ_\alpha - \alpha)^2}{(\rho^\circ_\alpha - 2\alpha)^2} A_2,$$
$$C_3 = \frac{\rho^\circ_\alpha - \alpha}{\rho^\circ_\alpha - 2\alpha} A_2, \quad C_4 = \frac{\rho^\circ_\alpha - \alpha}{\rho^\circ_\alpha - 3\alpha} A_2. \tag{43}$$

By (41) and (43), we have

$$\rho^\circ_\alpha - \alpha \geq \rho^\circ_\alpha - 2\alpha \geq \rho^\circ_\alpha - 3\alpha > \rho^\circ_\alpha - 4\alpha - A_2 > 0. \tag{44}$$

From (43) and (44), we get

$$C_1 > C_4 \geq C_3 \geq A_2 > 0, \quad C_2 \geq A_2 > 0. \tag{45}$$

By substituting (40), (42) and (43) into (38), we obtain

$$\rho^\circ_\alpha = (m-1)\alpha + (m-4)A_2 + 9C_1. \tag{46}$$

When $m \geq 20$, it follows from $A_2, C_1 > 0$, $0 \leq \alpha < 1$ and (46) that

$$\rho^\circ_\alpha = (m-1)\alpha + (m-4)A_2 + 9C_1 \geq 19\alpha + 16A_2 + 9C_1. \tag{47}$$

(1.1). The proof of $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

It is noted that $\mathcal{B}^{(1)}_{n,k}$ is shown in Fig. 3(c) with $a = m-2$ and $b = c = 0$. In $\mathcal{B}^{(1)}_{n,k}$, let $e_3, e_4, \cdots, e_m$ be the $m-2$ pendent edges attached at u_1. We construct a weighted incidence matrix $\boldsymbol{B}$ for $\mathcal{B}^{(1)}_{n,k}$ as follows. Let $B(v, e_i) = \rho^\circ_\alpha - \alpha$, where v is an arbitrary core vertex in $V\Big(\mathcal{B}^{(1)}_{n,k}\Big)$ and e_i $(1 \leq i \leq m)$ is the edge incident with v. Furthermore, let

$$B(u_1, e_1) = B(u_1, e_2) = 4C_2, \quad B(u_1, e_i) = A_2, \quad \text{where } 3 \leq i \leq m,$$
$$B(u_2, e_1) = B(u_2, e_2) = B(u_3, e_1) = B(u_3, e_2) = \frac{1}{2}\rho^\circ_\alpha - \alpha.$$

Since $A_2, C_2 > 0$ and $\rho^\circ_\alpha - 2\alpha > 0$ (by (44)), we get $B(v, e) > 0$ for any vertex v and any edge e incident with v in $\mathcal{B}^{(1)}_{n,k}$. We have $\prod_{v: v\in e} B(v, e) = (1-\alpha)^k$ for an arbitrary edge $e = e_i$ $(1 \leq i \leq m)$ in $E\Big(\mathcal{B}^{(1)}_{n,k}\Big)$. For any core vertex $v \in V\Big(\mathcal{B}^{(1)}_{n,k}\Big)$, $v = u_2$ and $v = u_3$, we have $\sum_{e: v\in e}\Big(B(v, e) + \alpha\Big) = \rho^\circ_\alpha$.

Next, we compare $\sum_{e: u_1\in e}\Big(B(u_1, e) + \alpha\Big)$ with ρ°_α. By (46), we get

$$\begin{aligned}
&\rho^\circ_\alpha - \sum_{e: u_1\in e}\Big(B(u_1, e) + \alpha\Big) = \rho^\circ_\alpha - \Big(2B(u_1, e_1) + (m-2)B(u_1, e_3) + m\alpha\Big)\\
&= -\alpha + 9C_1 - 8C_2 - 2A_2\\
&= -\alpha + D_1\Big[-(\rho^\circ_\alpha)^2(\rho^\circ_\alpha - 19\alpha - 10A_2) - \alpha^2(40\rho^\circ_\alpha - 28\alpha - 16A_2) - 24A_2\alpha\rho^\circ_\alpha\Big],
\end{aligned}$$

where $D_1 = \frac{A_2}{(\rho^\circ_\alpha - 4\alpha - A_2)(\rho^\circ_\alpha - 2\alpha)^2}$. Since $A_2 > 0$ and $\rho^\circ_\alpha - 2\alpha > \rho^\circ_\alpha - 4\alpha - A_2 > 0$ (by (44)), we get $D_1 > 0$. From (47), we have $\rho^\circ_\alpha - \sum_{e: u_1\in e}\Big(B(u_1, e) + \alpha\Big) < 0$. Next, we verify that B is consistent. In $\mathcal{B}^{(1)}_{n,k}$, for the three cycles $u_1e_1u_2e_2u_1$, $u_1e_1u_3e_2u_1$, and $u_2e_1u_3e_2u_2$, we can check $\frac{B(u_2,e_1)}{B(u_1,e_1)}\frac{B(u_1,e_2)}{B(u_2,e_2)} = 1$, $\frac{B(u_3,e_1)}{B(u_1,e_1)}\frac{B(u_1,e_2)}{B(u_3,e_2)} = 1$, and $\frac{B(u_3,e_1)}{B(u_2,e_1)}\frac{B(u_2,e_2)}{B(u_3,e_2)} = 1$, respectively. Therefore, $\mathcal{B}^{(1)}_{n,k}$ is strictly and consistently ρ°_α-supernormal. By Lemma 3.7, we obtain $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho^\circ_\alpha = \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

(1.2). The proof of $\rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{B}^{(2)}_{n,k}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

In $\mathcal{B}_{n,k}(a, b, c)$ (as shown in Fig. 3(c)), let $a = m-3$, $b = 1$ and $c = 0$ and we get $\mathcal{B}^{(2)}_{n,k}$. In $\mathcal{B}^{(2)}_{n,k}$, let v_1 and v_2 be the two core vertices which are respectively incident with e_1 and a pendent edge incident with u_1, where the vertices v_1, v_2, u_1, u_2, and u_3 of $\mathcal{B}^{(2)}_{n,k}$ and the edges e_1 and e_2 of $\mathcal{B}^{(2)}_{n,k}$ are shown in Fig. 3(c). In $\mathcal{B}^{(2)}_{n,k}$, let $e_3, e_4, \cdots, e_{m-1}$ be the $m-3$ pendent edges incident with u_1 and e_m be the pendent edge incident with u_2. We construct a weighted incidence matrix $\boldsymbol{B}$ for $\mathcal{B}^{(2)}_{n,k}$ as follows. Let $B(v, e_i) = \rho^\circ_\alpha - \alpha$, where v is an arbitrary core vertex in $V\Big(\mathcal{B}^{(2)}_{n,k}\Big)$ and e_i $(1 \leq i \leq m)$ is the edge incident with v. Furthermore, let

$$B(u_1, e_1) = B(u_1, e_2) = \frac{4(\rho^\circ_\alpha - 2\alpha)}{\rho^\circ_\alpha - 3\alpha - A_2}C_2, \qquad B(u_1, e_i) = A_2, \quad \text{where } 3 \leq i \leq m-1,$$
$$B(u_2, e_1) = B(u_2, e_2) = \frac{1}{2}(\rho^\circ_\alpha - 3\alpha - A_2), \qquad B(u_2, e_m) = A_2,$$
$$B(u_3, e_1) = B(u_3, e_2) = \frac{1}{2}(\rho^\circ_\alpha - 2\alpha).$$

Since $A_2, C_2 > 0$ and $\rho^\circ_\alpha - 2\alpha > \rho^\circ_\alpha - 3\alpha - A_2 \geq \rho^\circ_\alpha - 4\alpha - A_2 > 0$ (by (44)), we have $B(v, e) > 0$ for any vertex v and any edge e incident with v in $\mathcal{B}^{(2)}_{n,k}$. We get $\prod_{v: v\in e} B(v, e) = (1-\alpha)^k$, where $e = e_i$ $(1 \leq i \leq m)$ is an arbitrary edge in $E\Big(\mathcal{B}^{(2)}_{n,k}\Big)$. For any core vertex $v \in V\Big(\mathcal{B}^{(2)}_{n,k}\Big)$, $v = u_2$ and $v = u_3$, we have $\sum_{e: v\in e}\Big(B(v, e) + \alpha\Big) = \rho^\circ_\alpha$.

Next, we compare $\sum_{e:u_1\in e}\Big(B(u_1,e)+\alpha\Big)$ with ρ_α°. By (46), we have

$$\rho_\alpha^\circ-\sum_{e:u_1\in e}\Big(B(u_1,e)+\alpha\Big)=\rho_\alpha^\circ-\Big[2B(u_1,e_1)+(m-3)B(u_1,e_3)+(m-1)\alpha\Big]$$
$$=9C_1-\frac{8(\rho_\alpha^\circ-2\alpha)}{\rho_\alpha^\circ-3\alpha-A_2}C_2-A_2$$
$$=D_2(A_2\alpha+2\alpha^2)+D_3(3\alpha+A_2), \tag{48}$$

where $D_2=\frac{8(\rho_\alpha^\circ-\alpha)A_2}{(\rho_\alpha^\circ-2\alpha)(\rho_\alpha^\circ-3\alpha-A_2)(\rho_\alpha^\circ-4\alpha-A_2)}$ and $D_3=\frac{A_2}{\rho_\alpha^\circ-4\alpha-A_2}$. Owing to $A_2>0$ and $\rho_\alpha^\circ-\alpha\geq\rho_\alpha^\circ-2\alpha>\rho_\alpha^\circ-3\alpha-A_2\geq\rho_\alpha^\circ-4\alpha-A_2>0$ (by (44)), we get $D_2,D_3>0$. Therefore, it follows from $A_2,D_2,D_3>0$, $0\leq\alpha<1$ and (48) that $\rho_\alpha^\circ-\sum_{e:u_1\in e}\Big(B(u_1,e)+\alpha\Big)>0$. Hence, $\mathcal{B}_{n,k}^{(2)}$ is strictly ρ_α°-subnormal. By Lemma 3.5, we obtain $\rho_\alpha\Big(\mathcal{A}_{n,k}^{(2)}\Big)=\rho_\alpha^\circ>\rho_\alpha\Big(\mathcal{B}_{n,k}^{(2)}\Big)$ for $k\geq4$ and $m=\frac{n+1}{k-1}\geq20$.

(1.3). The proof of $\rho_\alpha\Big(\mathcal{A}_{n,k}^{(2)}\Big)>\rho_\alpha\Big(\mathcal{A}_{n,k}^{(3)}\Big)$ for $k\geq4$ and $m=\frac{n+1}{k-1}\geq20$.

In $\mathcal{A}'_{n,k}(a,b,c)$ (as shown in Fig. 3(a)), let $a=b=0$ and $c=m-3$ and we get $\mathcal{A}_{n,k}^{(3)}$. The vertices u_1, u_2 and u_3 of $\mathcal{A}_{n,k}^{(3)}$ and the edges e_1, e_2 and e_3 of $\mathcal{A}_{n,k}^{(3)}$ are shown in Fig. 3(a). In $\mathcal{A}_{n,k}^{(3)}$, let $e_4,e_5,\cdots,e_m$ be the $m-3$ pendent edges incident with u_3. For $\mathcal{A}_{n,k}^{(3)}$, we construct a weighted incidence matrix $\boldsymbol{B}$ as follows. Let $B(v,e_i)=\rho_\alpha^\circ-\alpha$, where v is an arbitrary core vertex in $V\Big(\mathcal{A}_{n,k}^{(3)}\Big)$ and e_i $(1\leq i\leq m)$ is the edge incident with v. Furthermore, let

$$B(u_1,e_1)=\rho_\alpha^\circ-3\alpha-6C_4,\quad B(u_1,e_2)=B(u_1,e_3)=3C_4,$$
$$B(u_2,e_1)=B(u_2,e_2)=B(u_2,e_3)=\frac{1}{3}(\rho_\alpha^\circ-3\alpha),$$
$$B(u_3,e_1)=\frac{3(\rho_\alpha^\circ-\alpha)}{\rho_\alpha^\circ-3\alpha-6C_4}C_4,\quad B(u_3,e_i)=A_2\quad \textit{where } 4\leq i\leq m.$$

From (43), we obtain $C_1=\frac{\rho_\alpha^\circ-\alpha}{\rho_\alpha^\circ-4\alpha-A_2}A_2>\frac{\rho_\alpha^\circ-\alpha}{\rho_\alpha^\circ-3\alpha}A_2=C_4>0$. It follows from (47) and $C_1>C_4$ that

$$\rho_\alpha^\circ-3\alpha-6C_4>\rho_\alpha^\circ-3\alpha-6C_1>16\alpha+16A_2+3C_1>0. \tag{49}$$

Since $C_4>0$, $\rho_\alpha^\circ-3\alpha>0$ (by (44)) and $\rho_\alpha^\circ-3\alpha-6C_4>0$, we obtain $B(v,e)>0$ for any vertex v and any edge e incident with v in $\mathcal{A}_{n,k}^{(3)}$. It is easy to check $\prod_{v:v\in e}B(v,e)=(1-\alpha)^k$ for an arbitrary edge $e=e_i$ $(1\leq i\leq m)$ in $E\Big(\mathcal{A}_{n,k}^{(3)}\Big)$. For any core vertex $v\in V\Big(\mathcal{A}_{n,k}^{(3)}\Big)$ and $v=u_1,u_2$, we have $\sum_{e:v\in e}\Big(B(v,e)+\alpha\Big)=\rho_\alpha^\circ$.

Next, we compare $\sum_{e:u_3\in e}\Big(B(u_3,e)+\alpha\Big)$ with ρ_α°. By (46), we have

$$\rho_\alpha^\circ-\sum_{e:u_3\in e}\Big(B(u_3,e)+\alpha\Big)=\rho_\alpha^\circ-\Big[B(u_3,e_1)+(m-3)B(u_3,e_4)+(m-2)\alpha\Big]$$
$$=\alpha+9C_1-\frac{3(\rho_\alpha^\circ-\alpha)}{\rho_\alpha^\circ-3\alpha-6C_4}C_4-A_2$$
$$>8C_4-\frac{3(\rho_\alpha^\circ-\alpha)}{\rho_\alpha^\circ-3\alpha-6C_4}C_4 \tag{50}$$
$$=\frac{5C_4}{\rho_\alpha^\circ-3\alpha-6C_4}\Big(\rho_\alpha^\circ-\frac{21}{5}\alpha-\frac{48}{5}C_4\Big).$$

It is noted that (50) follows from $C_1>C_4$ (by (43)) and $C_1>A_2$ (by (45)). Since $A_2,C_1>0$, from (47), we have $\rho_\alpha^\circ-19\alpha\geq0$. Therefore, $\frac{3}{2}A_2-C_4=\frac{A_2}{2(\rho_\alpha^\circ-3\alpha)}(\rho_\alpha^\circ-7\alpha)>0$. Namely, $\frac{3}{2}A_2>C_4$. Therefore, by (47), we have $\rho_\alpha^\circ \geq 19\alpha + 16A_2 + 9C_1 > 19\alpha + 16A_2 + 9C_4 > \frac{21}{5}\alpha + \frac{48}{5}C_4$. Furthermore, since $C_4 > 0$ (by (45)) and $\rho_\alpha^\circ - 3\alpha - 6C_4 > 0$ (by (49)), we have $\rho_\alpha^\circ > \sum_{e:u_3\in e}\Big(B(u_3, e) + \alpha\Big)$. Thus, $\mathcal{A}_{n,k}^{(3)}$ is strictly ρ_α°-subnormal. By Lemma 3.5, we obtain $\rho_\alpha\Big(\mathcal{A}_{n,k}^{(2)}\Big) = \rho_\alpha^\circ > \rho_\alpha\Big(\mathcal{A}_{n,k}^{(3)}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

(1.4). The proof of $\rho_\alpha\Big(\mathcal{A}_{n,k}^{(2)}\Big) > \rho_\alpha\Big(\mathcal{B}_{n,k}^{(3)}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

In $\mathcal{B}_{n,k}^{(3)}$, the vertices u_1, u_2, u_3, and u_4 and the edges e_1 and e_2 are shown in Fig. 3(c). In $\mathcal{B}_{n,k}^{(3)}$, let $e_3, e_4, \cdots, e_m$ be the $m-2$ pendent edges incident with u_4. For $\mathcal{B}_{n,k}^{(3)}$, we construct a weighted incidence matrix $\boldsymbol{B}$ as follows. Let $B(v, e_i) = \rho_\alpha^\circ - \alpha$, where v is an arbitrary core vertex in $V\Big(\mathcal{B}_{n,k}^{(3)}\Big)$ and e_i $(1 \leq i \leq m)$ is the edge incident with v. Furthermore, let

$$B(u_1, e_1) = \rho_\alpha^\circ - 2\alpha - 4C_2, \quad B(u_1, e_2) = 4C_2,$$

$$B(u_2, e_1) = B(u_2, e_2) = B(u_3, e_1) = B(u_3, e_2) = \frac{1}{2}(\rho_\alpha^\circ - 2\alpha),$$

$$B(u_4, e_1) = \frac{4(\rho_\alpha^\circ - \alpha)}{\rho_\alpha^\circ - 2\alpha - 4C_2}C_2, \quad B(u_4, e_i) = A_2, \quad \text{where } 3 \leq i \leq m.$$

Since $0 \leq \alpha < 1$, $\rho_\alpha^\circ - \alpha > 0$ and $A_2 > 0$, we get $(\rho_\alpha^\circ - 2\alpha)^2 > (\rho_\alpha^\circ - \alpha)(\rho_\alpha^\circ - 4\alpha - A_2)$. Therefore, we obtain

$$C_1 = \frac{\rho_\alpha^\circ - \alpha}{\rho_\alpha^\circ - 4\alpha - A_2}A_2 > \frac{(\rho_\alpha^\circ - \alpha)^2}{(\rho_\alpha^\circ - 2\alpha)^2}A_2 = C_2. \tag{51}$$

Thus, by (47), we have

$$\rho_\alpha^\circ - 2\alpha - 4C_2 > \rho_\alpha^\circ - 2\alpha - 4C_1 \geq 17\alpha + 16A_2 + 5C_1 > 0. \tag{52}$$

Since $A_2, C_2 > 0$, $\rho_\alpha^\circ - \alpha, \rho_\alpha^\circ - 2\alpha > 0$ (by (44)) and $\rho_\alpha^\circ - 2\alpha - 4C_2 > 0$, we have $B(v, e) > 0$ for any vertex v and any edge e incident with v in $\mathcal{B}_{n,k}^{(3)}$. We can check $\prod_{v:v\in e} B(v, e) = (1-\alpha)^k$ for any $e = e_i$ $(1 \leq i \leq m)$ in $E\Big(\mathcal{B}_{n,k}^{(3)}\Big)$. For any core vertex $v \in V\Big(\mathcal{B}_{n,k}^{(3)}\Big)$ and $v = u_1, u_2, u_3$, we have $\sum_{e:v\in e}\Big(B(v, e) + \alpha\Big) = \rho_\alpha^\circ$.

Next, we compare $\sum_{e:u_4\in e}\Big(B(u_4, e) + \alpha\Big)$ with ρ_α°. By (46), we obtain

$$\begin{aligned}
&\rho_\alpha^\circ - \sum_{e:u_4\in e}\Big(B(u_4, e) + \alpha\Big) \\
&= \rho_\alpha^\circ - \Big[B(u_4, e_1) + (m-2)B(u_4, e_3) + (m-1)\alpha\Big] \\
&= 9C_1 - \frac{4(\rho_\alpha^\circ - \alpha)}{\rho_\alpha^\circ - 2\alpha - 4C_2}C_2 - 2A_2 \\
&\geq 9C_2 - \frac{6(\rho_\alpha^\circ - \alpha)}{\rho_\alpha^\circ - 2\alpha - 4C_2}C_2 \qquad (53) \\
&= \frac{3C_2}{\rho_\alpha^\circ - 2\alpha - 4C_2}(\rho_\alpha^\circ - 4\alpha - 12C_2). \qquad (54)
\end{aligned}$$

It is noted that (53) follows from $C_1 > C_2$ (by (51)) and $\frac{(\rho_\alpha^\circ-\alpha)}{\rho_\alpha^\circ-2\alpha-4C_2}C_2 = \frac{(\rho_\alpha^\circ-\alpha)^3}{(\rho_\alpha^\circ-2\alpha)^2(\rho_\alpha^\circ-2\alpha-4C_2)}A_2 > A_2$. Since $C_1 > 0$, by (47), we get

$$\frac{3}{2}A_2 - C_2 = \frac{A_2}{(\rho_\alpha^\circ - 2\alpha)^2}\left(\frac{1}{2}\rho_\alpha^\circ(\rho_\alpha^\circ - 8\alpha) + 5\alpha^2\right) > 0.$$

Namely, $\frac{3}{2}A_2 > C_2$. It follows from $C_1 > C_2$ (by (51)), $\frac{3}{2}A_2 > C_2$ and (47) that $\rho_\alpha^\circ - 4\alpha - 12C_2 \geq 15\alpha + \frac{23}{2}A_2 > 0$. Furthermore, since $C_2 > 0$ and $\rho_\alpha^\circ - 2\alpha - 4C_2 > 0$ (by (52)), by (54), we get $\rho_\alpha^\circ > \sum_{e:u_4\in e}\Big(B(u_4, e) + \alpha\Big)$. Thus, $\mathcal{B}_{n,k}^{(3)}$ is strictly ρ_α°-subnormal. By Lemma 3.5, we obtain $\rho_\alpha\Big(\mathcal{A}_{n,k}^{(2)}\Big) = \rho_\alpha^\circ > \rho_\alpha\Big(\mathcal{B}_{n,k}^{(3)}\Big)$ for $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. □

5. The hypergraphs with the first and the second largest α-spectral radii among $\mathcal{B}(n,k)$

In this section, we will characterize the hypergraphs with the first and the second largest α-spectral radii among $\mathcal{B}(n,k)$. To obtain our results, Lemmas 5.1–5.8 are needed.

Lemma 5.1. *We have* $\rho_\alpha\Big(\mathcal{A}_{n,k}(a+1,b-1)\Big) > \rho_\alpha\Big(\mathcal{A}_{n,k}(a,b)\Big)$, *where* $k \geq 3$, $a \geq b \geq 1$ *and* $a+b = m-3$.

Proof. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ be the α-Perron vector of $\rho_\alpha\Big(\mathcal{A}_{n,k}(a,b)\Big)$. If $x_{u_1} \geq x_{u_2}$, in $\mathcal{A}_{n,k}(a,b)$, by removing one pendent edge from u_2 to u_1, we get $\mathcal{A}_{n,k}(a+1,b-1)$. By Lemma 2.6, we get Lemma 5.1. If $x_{u_2} > x_{u_1}$, in $\mathcal{A}_{n,k}(a,b)$, by removing $a-b+1$ pendent edges from u_1 to u_2, we obtain $\mathcal{A}_{n,k}(a+1,b-1)$. By Lemma 2.6, we also have Lemma 5.1. □

Corollary 5.2. *We have* $\rho_\alpha\Big(\mathcal{A}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big) \geq \rho_\alpha\Big(\mathcal{A}_{n,k}(a,b)\Big)$ *with the equality if and only if* $\mathcal{A}_{n,k}(a,b) = \mathcal{A}^{(2)}_{n,k}$, *where* $k \geq 3$, $a \geq b \geq 1$ *and* $a+b = m-3$.

Proof. By repeatedly using Lemma 5.1 and bearing the definitions of $\mathcal{A}^{(1)}_{n,k}$ and $\mathcal{A}^{(2)}_{n,k}$ in mind, we get Corollary 5.2. □

By the methods similar to those for Lemma 5.1, we have Lemma 5.3 as follows.

Lemma 5.3. *We have* $\rho_\alpha\Big(\mathcal{B}_{n,k}(a+1,b-1,0)\Big) > \rho_\alpha\Big(\mathcal{B}_{n,k}(a,b,0)\Big)$, *where* $k \geq 4$, $a \geq b \geq 1$ *and* $a+b = m-2$.

By the methods similar to those for Corollary 5.2, we obtain Corollary 5.4.

Corollary 5.4. *We have* $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{B}^{(2)}_{n,k}\Big) \geq \rho_\alpha\Big(\mathcal{B}_{n,k}(a,b,0)\Big)$ *with the equality if and only if* $\mathcal{B}_{n,k}(a,b,0) = \mathcal{B}^{(2)}_{n,k}$, *where* $k \geq 4$, $a \geq b \geq 1$ *and* $a+b = m-2$.

Lemma 5.5. *Let* $\mathcal{H} \in \mathcal{B}_3(n,k) \setminus \{\mathcal{B}^{(1)}_{n,k}, C_{n,k}, \mathcal{F}_{n,k}\}$, *where* $k \geq 4$ *and* $m = \frac{n-1}{k-1} \geq 20$. *We have* $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > max\{\rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big), \rho_\alpha\Big(C_{n,k}\Big), \rho_\alpha\Big(\mathcal{F}_{n,k}\Big)\} > \rho_\alpha(\mathcal{H})$, *where* $0 \leq \alpha < 1$.

Proof. Let $k \geq 4$ and $m = \frac{n-1}{k-1} \geq 20$. Six cases are considered as follows.

Case (1). $\mathcal{H} = \mathcal{A}'_{n,k}(a,b,c)$.

By the definition of $\mathcal{A}'_{n,k}(a,b,c)$, we have $c \geq 1$. Let $a \geq b$. If $a = b = 0$, then $\mathcal{H} = \mathcal{A}^{(3)}_{n,k}$. By Lemma 4.3, we have $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(3)}_{n,k}\Big)$. Namely, Lemma 5.5 holds when $a = b = 0$. Next, let $a \geq 1$. In $\mathcal{A}'_{n,k}(a,b,c)$, if $x_{u_2} \geq x_{u_3}$, then by removing all the edges incident with u_3 from u_3 to u_2, we get $\mathcal{A}_{n,k}(a,b+c)$, where $a \geq 1$ and $b+c \geq 1$. By Lemma 2.6, we have $\rho_\alpha\Big(\mathcal{A}_{n,k}(a,b+c)\Big) > \rho_\alpha\Big(\mathcal{A}'_{n,k}(a,b,c)\Big)$. In $\mathcal{A}'_{n,k}(a,b,c)$, if $x_{u_2} < x_{u_3}$, then by removing all the edges incident with u_2 (except for e_1) from u_2 to u_3, we get $\mathcal{A}_{n,k}(a,b+c)$, where $a \geq 1$ and $b+c \geq 1$. By Lemma 2.6, we obtain $\rho_\alpha\Big(\mathcal{A}_{n,k}(a,b+c)\Big) > \rho_\alpha\Big(\mathcal{A}'_{n,k}(a,b,c)\Big)$. Since $a \geq 1$ and $b+c \geq 1$, by Corollary 5.2, we obtain $\rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big) \geq \rho_\alpha\Big(\mathcal{A}_{n,k}(a,b+c)\Big)$. By Lemma 4.3, we have $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big)$. Thus, we get $\rho_\alpha\Big(\mathcal{B}^{(1)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}^{(2)}_{n,k}\Big) > \rho_\alpha\Big(\mathcal{A}'_{n,k}(a,b,c)\Big)$ with $a \geq 1$.

Case (2). $\mathcal{H} = \mathcal{A}^{*}_{n,k}(a,b,c)$.

By the definition of $\mathcal{A}^{*}_{n,k}(a,b,c)$, we have $c \geq 1$. In $\mathcal{A}^{*}_{n,k}(a,b,c)$, if $x_{u_2} \geq x_{u_3}$, then by removing all the pendent edges incident with u_3 from u_3 to u_2, we get $\mathcal{A}_{n,k}(a+1,b+c)$, where $a \geq 0$ and $b+c \geq 1$. By Lemma 2.6, we obtain $\rho_\alpha\Big(\mathcal{A}_{n,k}(a+1,b+c)\Big) > \rho_\alpha\Big(\mathcal{A}^{*}_{n,k}(a,b,c)\Big)$. In $\mathcal{A}^{*}_{n,k}(a,b,c)$, if $x_{u_2} < x_{u_3}$, by removing all the edges incident with u_2 (except for e_1) from u_2 to u_3, we also obtain $\mathcal{A}_{n,k}(a+1,b+c)$, where $a \geq 0$ and $b+c \geq 1$. By Lemma 2.6, we get $\rho_\alpha\Big(\mathcal{A}_{n,k}(a+1,b+c)\Big) > \rho_\alpha\Big(\mathcal{A}^{*}_{n,k}(a,b,c)\Big)$. Furthermore, by the methods similar to those for the proofs of Case (1), we get Lemma 5.5.

Case (3). $\mathcal{H} = \mathcal{B}_{n,k}(a,b,c)$ and $\mathcal{H} \neq \mathcal{B}^{(1)}_{n,k}$.

In $\mathcal{B}_{n,k}(a,b,c)$, without loss of generality, let $a \geq b \geq c$. Since $\mathcal{H} \neq \mathcal{B}^{(1)}_{n,k}$, at least two of a, b and c are nonzero. Two subcases are considered as follows.

Subcase (3.1). $c = 0$.

Since $\mathcal{H} \neq \mathcal{B}_{n,k}^{(1)}$, we have $b \geq 1$. If $b = 1$ and $c = 0$, then $\mathcal{H} = \mathcal{B}_{n,k}^{(2)}$. By Lemma 4.3, we have $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{B}_{n,k}^{(2)}\right)$. Namely, Lemma 5.5 holds when $b = 1$ and $c = 0$. Let $b \geq 2$. Since $a \geq b$, we have $a \geq 2$. In $\mathcal{B}_{n,k}(a, b, c)$, by the symmetry, without loss of generality, let $x_{u_1} \geq x_{u_2}$. We remove the $b - 1$ pendent edges incident with u_2 from u_2 to u_1 and get $\mathcal{B}_{n,k}^{(2)}$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{B}_{n,k}^{(2)}\right) > \rho_\alpha\left(\mathcal{B}_{n,k}(a, b, c)\right)$. Furthermore, it follows from Lemma 4.3 that $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right) > \rho_\alpha\left(\mathcal{B}_{n,k}^{(2)}\right) > \rho_\alpha\left(\mathcal{B}_{n,k}(a, b, c)\right)$ when $\mathcal{B}_{n,k}(a, b, c) \neq \mathcal{B}_{n,k}^{(1)}, \mathcal{B}_{n,k}^{(2)}$ and $c = 0$.

Subcase (3.2). $c \geq 1$.

In this subcase, we have $a \geq b \geq c \geq 1$. In $\mathcal{B}_{n,k}(a, b, c)$, by the symmetry, without loss of generality, we assume $x_{u_1} \geq x_{u_3}$. We remove the c pendent edges incident with u_3 from u_3 to u_1 and get $\mathcal{B}_{n,k}(a+c, b, 0)$, where $a + c \geq 2$ and $b \geq 1$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{B}_{n,k}(a + c, b, 0)\right) > \rho_\alpha\left(\mathcal{B}_{n,k}(a, b, c)\right)$. Furthermore, it follows from Lemma 4.3 and Corollary 5.4 that $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right) > \rho_\alpha\left(\mathcal{B}_{n,k}^{(2)}\right) \geq \rho_\alpha\left(\mathcal{B}_{n,k}(a+c, b, 0)\right) > \rho_\alpha\left(\mathcal{B}_{n,k}(a, b, c)\right)$.

Case (4). $\mathcal{H} = C_{n,k}(a, b, c)$ and $\mathcal{H} \neq C_{n,k}$.

In $C_{n,k}(a, b, c)$, we assume $b \geq c$. Since $\mathcal{H} \neq C_{n,k}$, we have $b \geq 1$.

In $C_{n,k}(a, b, c)$, if $x_{u_2} \geq x_{u_3}$, by removing all the edges incident with u_3 (except for e_3) from u_3 to u_2, we get $\mathcal{A}_{n,k}(a + 1, b + c)$, where $a \geq 0$ and $b + c \geq 1$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{A}_{n,k}(a + 1, b + c)\right) > \rho_\alpha\left(C_{n,k}(a, b, c)\right)$. Similarly, if $x_{u_2} < x_{u_3}$, we also get $\rho_\alpha\left(\mathcal{A}_{n,k}(a + 1, b + c)\right) > \rho_\alpha\left(C_{n,k}(a, b, c)\right)$. Since $a + 1, b + c \geq 1$, by Corollary 5.2, we have $\rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right) \geq \rho_\alpha\left(\mathcal{A}_{n,k}(a + 1, b + c)\right)$. By Lemma 4.3, we obtain $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right)$. Thus, we get $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right) > \rho_\alpha\left(C_{n,k}(a, b, c)\right)$.

Case (5). $\mathcal{H} = \mathcal{D}_{n,k}(a, b, c)$.

In $\mathcal{D}_{n,k}(a, b, c)$, if $x_{u_1} \geq x_{u_2}$, by removing e_2 from u_2 to u_1, we obtain $C_{n,k}(a, b, c)$. By Lemma 2.6, we get $\rho_\alpha\left(C_{n,k}(a, b, c)\right) > \rho_\alpha\left(\mathcal{D}_{n,k}(a, b, c)\right)$. Similarly, if $x_{u_1} < x_{u_2}$, by removing e_1 from u_1 to u_2, we obtain $\rho_\alpha\left(C_{n,k}(b, a, c)\right) > \rho_\alpha\left(\mathcal{D}_{n,k}(a, b, c)\right)$. If $C_{n,k}(a, b, c) = C_{n,k}$ or $C_{n,k}(b, a, c) = C_{n,k}$, then by Lemma 4.2, we have $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(C_{n,k}\right)$. Thus, we get $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(C_{n,k}\right) > \rho_\alpha\left(\mathcal{D}_{n,k}(a, b, c)\right)$. If $C_{n,k}(a, b, c), C_{n,k}(b, a, c) \neq C_{n,k}$, then by the proofs of Case (4), we obtain $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{A}_{n,k}^{(2)}\right) > \rho_\alpha\left(\mathcal{D}_{n,k}(a, b, c)\right)$.

Case (6). $\mathcal{H} = \mathcal{F}_{n,k}(a, b, c)$ and $\mathcal{H} \neq \mathcal{F}_{n,k}$.

In this case, since $\mathcal{H} = \mathcal{F}_{n,k}(a, b, c)$ and $\mathcal{H} \neq \mathcal{F}_{n,k}$, we have $b \geq 1$ or $c \geq 1$. Two subcases are considered as follows.

Subcase (6.1). $b \geq 1$.

In $\mathcal{F}_{n,k}(a, b, c)$, if $x_{u_1} \geq x_{u_2}$, we remove all the b pendent edges incident with u_2 from u_2 to u_1 and get $\mathcal{F}_{n,k}(a + b, 0, c)$, where $a + b \geq 1$ and $c \geq 0$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{F}_{n,k}(a + b, 0, c)\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a, b, c)\right)$. In $\mathcal{F}_{n,k}(a, b, c)$, if $x_{u_1} < x_{u_2}$, we remove all the edges incident with u_1 (except for e_1 and e_2) from u_1 to u_2 and obtain $\mathcal{F}_{n,k}(a+b, 0, c)$, where $a+b \geq 1$ and $c \geq 0$. It follows from Lemma 2.6 that $\rho_\alpha\left(\mathcal{F}_{n,k}(a+b, 0, c)\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a, b, c)\right)$.

If $c = 0$, then $\mathcal{F}_{n,k}(a + b, 0, c) = \mathcal{F}_{n,k}$. By Lemma 4.2, we get $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right)$. Thus, we have $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a, b, c)\right)$. Namely, Lemma 5.5 holds when $\mathcal{H} = \mathcal{F}_{n,k}(a, b, c)$ with $b \geq 1$ and $c = 0$.

Let $c \geq 1$. In $\mathcal{F}_{n,k}(a + b, 0, c)$, if $x_{u_1} \geq x_{u_3}$, we remove all the c pendent edges incident with u_3 from u_3 to u_1 and get $\mathcal{F}_{n,k}$. By Lemma 2.6, we get $\rho_\alpha\left(\mathcal{F}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a + b, 0, c)\right)$. In $\mathcal{F}_{n,k}(a + b, 0, c)$, if $x_{u_1} < x_{u_3}$, we remove all the edges incident with u_1 (except for e_1 and e_3) from u_1 to u_3 and obtain $\mathcal{F}_{n,k}$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{F}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a + b, 0, c)\right)$. By Lemma 4.2, we get $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right)$. Thus, we have $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a + b, 0, c)\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a, b, c)\right)$ with $b \geq 1$ and $c \geq 1$. Namely, Lemma 5.5 holds when $\mathcal{H} = \mathcal{F}_{n,k}(a, b, c)$ with $b, c \geq 1$.

Subcase (6.2). $b = 0$.

In this subcase, we have $c \geq 1$. By the methods similar to those for the proofs of Subcase (6.1), we get $\rho_\alpha\left(\mathcal{B}_{n,k}^{(1)}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}\right) > \rho_\alpha\left(\mathcal{F}_{n,k}(a, b, c)\right)$ with $b = 0$ and $c \geq 1$.

By combining the proofs of Cases (1)–(6), we get Lemma 5.5. □

Lemma 5.6. *Let $\mathcal{H} \in \mathcal{B}_i(n,k)$, where $i \geq 4$, $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 5$. Suppose that all the IVs of $\mathcal{H}$ are incident with one edge f in $E(\mathcal{H})$. Then, in $\mathcal{H}$, there exist two IVs, denoted by u_{k_1} and u_{k_2} $(1 \leq k_1 < k_2 \leq i)$, except for f, such that there does not exist another edge satisfying that u_{k_1} and u_{k_2} are incident with this edge simultaneously.*

Proof. We suppose that Lemma 5.6 do not hold. Namely, for any two IVs u_{i_1} and u_{i_2} in $\mathcal{H}$, there exists another edge (denoted by e^*, $e^* \neq f$) such that $u_{i_1}, u_{i_2} \in e^*$, where $1 \leq i_1 < i_2 \leq i$. Since $i \geq 4$, $\mathcal{H}$ contains a 3-cyclic hypergraph as its subhypergraph. By Lemma 2.7, the number of cyclomatics of $\mathcal{H}$ is not less than 3. This contradicts the fact that $\mathcal{H}$ is a 2-cyclic hypergraph. □

Lemma 5.7. *Let $\mathcal{H} \in \mathcal{B}_4(n,k)$, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. We have $max\{\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right), \rho_\alpha\left(C_{n,k}\right), \rho_\alpha\left(\mathcal{F}_{n,k}\right)\} > \rho_\alpha(\mathcal{H})$.*

Proof. Two cases are considered as follows.

Case (1). $\mathcal{H}$ has exactly two non-pendent edges (denoted by e_1 and e_2).

Since $\mathcal{H}$ is a bicyclic hypergraph, we have $|e_1 \bigcap e_2| = 3$. Let $e_1 \bigcap e_2 = \{u_1, u_2, u_3\}$. Since $\mathcal{H} \in \mathcal{B}_4(n,k)$, $\mathcal{H}$ has four IVs. Thus, in $\mathcal{H}$, there exists another IV (denoted by u_4) such that u_4 is incident with e_1 or e_2, say e_1. Obviously, $\mathcal{H}$ is a hypergraph obtained from $\mathcal{B}_{n,k}(a,b,c)$ by attaching d pendent edges at u_4, where $d \geq 1$. Without loss of generality, we suppose $a \geq b \geq c$.

If $a = 0$, then $b = c = 0$. Namely, $\mathcal{H} = \mathcal{B}^{(3)}_{n,k}$. By Lemma 4.3, we obtain $\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right) > \rho_\alpha\left(\mathcal{B}^{(3)}_{n,k}\right)$.

Next, let $a \geq 1$. In $\mathcal{H}$, if $x_{u_2} \geq x_{u_4}$, then by removing all the d pendent edges incident with u_4 from u_4 to u_2, we obtain $\mathcal{B}_{n,k}(a, b+d, c)$, where $b+d \geq 1$ and $c \geq 0$. By Lemma 2.6, we have $\rho_\alpha\left(\mathcal{B}_{n,k}(a, b+d, c)\right) > \rho_\alpha(\mathcal{H})$. In $\mathcal{H}$, if $x_{u_2} < x_{u_4}$, by removing all the edges incident with u_2 (except for e_1) from u_2 to u_4, we get $\mathcal{B}_{n,k}(a, b+d, c)$, where $b+d \geq 1$ and $c \geq 0$. By Lemma 2.6, we also have $\rho_\alpha\left(\mathcal{B}_{n,k}(a, b+d, c)\right) > \rho_\alpha(\mathcal{H})$. Since $a, b+d \geq 1$, by Corollary 5.4, we have $\rho_\alpha\left(\mathcal{B}^{(2)}_{n,k}\right) \geq \rho_\alpha\left(\mathcal{B}_{n,k}(a, b+d, c)\right)$. By Lemma 4.3, we get $\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right) > \rho_\alpha\left(\mathcal{B}^{(2)}_{n,k}\right)$. Thus, we obtain $\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right) > \rho_\alpha\left(\mathcal{B}^{(2)}_{n,k}\right) \geq \rho_\alpha\left(\mathcal{B}_{n,k}(a, b+d, c)\right) > \rho_\alpha(\mathcal{H})$.

Case (2). $\mathcal{H}$ has at least three non-pendent edges.

Subcase (2.1). All the IVs of $\mathcal{H}$ are incident with one edge (denoted by f).

By Lemma 5.6, in $\mathcal{H}$, there exist two IVs (denoted by v_1 and v_2) such that there does not exist an edge in $E(\mathcal{H}) \backslash f$ satisfying that v_1 and v_2 are incident with this edge simultaneously. Suppose $x_{v_1} \geq x_{v_2}$. By moving all the edges incident with v_2 (except for f) from v_2 to v_1, we obtain a hypergraph (denoted by $\mathcal{H}'$). Obviously, $\mathcal{H}' \in \mathcal{B}_3(n,k)$. By Lemma 2.6, we get $\rho_\alpha(\mathcal{H}') > \rho_\alpha(\mathcal{H})$. It is noted that $V(\mathcal{H}) = V(\mathcal{H}')$, $d_{\mathcal{H}'}(v_2) = 1 < d_{\mathcal{H}}(v_2)$, $d_{\mathcal{H}'}(v_1) > d_{\mathcal{H}}(v_1)$, and $d_{\mathcal{H}'}(u) = d_{\mathcal{H}}(u)$ for $u \in V(\mathcal{H}') \setminus \{v_1, v_2\}$. In $\mathcal{H}'$, since f has three IVs and f is a non-pendent edge, $\mathcal{H}'$ and $\mathcal{H}$ have the same number of non-pendent edges. Namely, $\mathcal{H}'$ has at least three non-pendent edges. Obviously, $\mathcal{H}' \neq \mathcal{B}^{(1)}_{n,k}$ since $\mathcal{B}^{(1)}_{n,k}$ has only two non-pendent edges. By Lemma 5.5, we have $\max\{\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right), \rho_\alpha\left(C_{n,k}\right), \rho_\alpha\left(\mathcal{F}_{n,k}\right)\} \geq \rho_\alpha(\mathcal{H}')$. Thus, we get $\max\{\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right), \rho_\alpha\left(C_{n,k}\right), \rho_\alpha\left(\mathcal{F}_{n,k}\right)\} \geq \rho_\alpha(\mathcal{H}') > \rho_\alpha(\mathcal{H})$.

Subcase (2.2). In $\mathcal{H}$, there does not exist an edge such that it is incident with all the IVs of $\mathcal{H}$.

In this case, in $\mathcal{H}$, there exist two IVs, denoted by u_1 and u_2, such that they are not incident with a common edge. Otherwise, in $\mathcal{H}$, if any two IVs are incident with a common edge, then $\mathcal{H}$ contains a 3-cyclic hypergraph as its subhypergraph. This is a contradiction. Let $P = u_1 e_1 \cdots e_s u_2$ be the shortest path connecting u_1 and u_2, where $s \geq 2$. In $\mathcal{H}$, if $x_{u_2} \geq x_{u_1}$, let $\mathcal{H}^\circ$ be the k-uniform hypergraph obtained from $\mathcal{H}$ by removing all the edges incident with u_1 (except for e_1) from u_1 to u_2. Since $d_{\mathcal{H}^\circ}(u_1) = 1$, we have $\mathcal{H}^\circ \in \mathcal{B}_3(n,k)$. By Lemma 2.6, we have $\rho_\alpha(\mathcal{H}^\circ) > \rho_\alpha(\mathcal{H})$. In $\mathcal{H}$, if $x_{u_2} < x_{u_1}$, let $\mathcal{H}^\Delta$ be the k-uniform hypergraph obtained from $\mathcal{H}$ by removing all the edges incident with u_2 (except for e_s) from u_2 to u_1. Since $d_{\mathcal{H}^\Delta}(u_2) = 1$, we have $\mathcal{H}^\Delta \in \mathcal{B}_3(n,k)$. By Lemma 2.6, we have $\rho_\alpha(\mathcal{H}^\Delta) > \rho_\alpha(\mathcal{H})$. Next, we prove $\mathcal{H}^\circ$, $\mathcal{H}^\Delta \neq \mathcal{B}^{(1)}_{n,k}$.

In $\mathcal{H}$, if at least one of u_1 and u_2 is incident with pendent edges, then by the definition of $\mathcal{H}^\circ$, there exists a pendent edge incident with u_2 in $\mathcal{H}^\circ$. Thus, in $\mathcal{H}^\circ$, the shortest path connecting u_1 and an arbitrary pendent vertex incident with a pendent edge attached at u_2 is at least of length 3. This implies that $\mathcal{H}^\circ \neq \mathcal{B}^{(1)}_{n,k}$ since the diameter of $\mathcal{B}^{(1)}_{n,k}$ is 2. Similarly, we have $\mathcal{H}^\Delta \neq \mathcal{B}^{(1)}_{n,k}$.

Next, In $\mathcal{H}$, we suppose that both of u_1 and u_2 are not incident with pendent edges. Since u_1 and u_2 are two IVs, u_1 is incident with a non-pendent edge (denoted by f_1, $f_1 \neq e_1$) and u_2 is incident with a non-pendent edge (denoted by f_2, $f_1 \neq e_2$). By the definition of $\mathcal{H}^\circ$, in $\mathcal{H}^\circ$, there are three non-pendent edges, namely $(f_1 \setminus \{u_1\}) \cup \{u_2\}$, f_2 and e_s. Thus, we get $\mathcal{H}^\circ \neq \mathcal{B}^{(1)}_{n,k}$ since $\mathcal{B}^{(1)}_{n,k}$ has only two non-pendent edges. Similarly, we have $\mathcal{H}^\Delta \neq \mathcal{B}^{(1)}_{n,k}$.

By the above proofs, we have $\mathcal{H}^\circ, \mathcal{H}^\Delta \in \mathcal{B}_3(n,k)$ and $\mathcal{H}^\circ, \mathcal{H}^\Delta \neq \mathcal{B}^{(1)}_{n,k}$. Thus, by Lemma 5.5, we have $\max\{\rho_\alpha(\mathcal{A}^{(2)}_{n,k}), \rho_\alpha(C_{n,k}), \rho_\alpha(\mathcal{F}_{n,k})\} \geq \max\{\rho_\alpha(\mathcal{H}^\circ), \rho_\alpha(\mathcal{H}^\Delta)\}$. Therefore, we have $\max\{\rho_\alpha(\mathcal{A}^{(2)}_{n,k}), \rho_\alpha(C_{n,k}), \rho_\alpha(\mathcal{F}_{n,k})\} > \rho_\alpha(\mathcal{H})$. □

Lemma 5.8. *Let $\mathcal{H} \in \mathcal{B}_i(n,k)$, where $i \geq 4$, $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. We have $\max\{\rho_\alpha(\mathcal{A}^{(2)}_{n,k}), \rho_\alpha(C_{n,k}), \rho_\alpha(\mathcal{F}_{n,k})\} > \rho_\alpha(\mathcal{H})$.*

Proof. Let $U = \{u_1, u_2, \cdots, u_i\}$ be the set of all the IVs of $\mathcal{H}$, where $i \geq 4$. We prove Claim (1) firstly.

Claim (1): For $\mathcal{H} \in \mathcal{B}_i(n,k)$ with $i \geq 4$ and $k \geq 4$, there exists a hypergraph $\mathcal{H}^\triangleright \in \mathcal{B}_{i-1}(n,k)$ such that $\rho_\alpha(\mathcal{H}^\triangleright) > \rho_\alpha(\mathcal{H})$, where $0 \leq \alpha < 1$.

To obtain Claim (1), two cases are considered as follows.

Case (1). In $\mathcal{H}$, there exists an edge (denoted by f) such that $U \subseteq f$.

By Lemma 5.6, in $\mathcal{H}$, there exist two IVs, denoted by u_{k_1} and u_{k_2} $(1 \leq k_1 < k_2 \leq i)$, except for f, such that there does not exist another edge satisfying that u_{k_1} and u_{k_2} are incident with this edge simultaneously. Without loss of generality, we suppose $x_{u_{k_1}} \geq x_{u_{k_2}}$. Let $\mathcal{H}^\triangleright$ be the hypergraph obtained from $\mathcal{H}$ by removing all the edges incident with u_{k_2} (except for f) from u_{k_2} to u_{k_1}. Obviously, $\mathcal{H}^\triangleright \in \mathcal{B}_{i-1}(n,k)$. By Lemma 2.6, we get $\rho_\alpha(\mathcal{H}^\triangleright) > \rho_\alpha(\mathcal{H})$.

Case (2). In $\mathcal{H}$, there does not exist an edge such that all the vertices in U are incident with it.

In this case, in $\mathcal{H}$, we claim that there exist two vertices u_{k_1} and u_{k_2} $(1 \leq k_1 < k_2 \leq i)$ in U in such a way that there does not exist an edge satisfying that u_{k_1} and u_{k_2} are incident with this edge simultaneously. Otherwise, we suppose that, in U, for any two vertices u_{i_1} and u_{i_2} $(1 \leq i_1 < i_2 \leq i)$, there exists an edge (denoted by e) satisfying that $u_{i_1}, u_{i_2} \in e$, where $e \in E(\mathcal{H})$. Since $i \geq 4$, $\mathcal{H}$ contains a 3-cyclic hypergraph as its subhypergraph. Since $\mathcal{H} \in \mathcal{B}_i(n,k)$, where $k \geq 4$ and $i \geq 4$, by Lemma 2.7, the number of cyclomatics of $\mathcal{H}$ is not less than 3. This contradicts the fact that $\mathcal{H}$ is a 2-cyclic hypergraph. Since $\mathcal{H}$ is connected, there exists one shortest path connecting u_{k_1} and u_{k_2}. We denote this path by $v_1e_1v_2 \dots e_hv_{h+1}$, where $h \geq 2$, $v_1 = u_{k_1}$ and $v_{h+1} = u_{k_2}$. Without loss of generality, we suppose $x_{u_{k_1}} \geq x_{u_{k_2}}$. Let $\mathcal{H}^\star$ be the hypergraph obtained from $\mathcal{H}$ by removing all the edges incident with u_{k_2} (except for e_h) from u_{k_2} to u_{k_1}. Obviously, $\mathcal{H}^\star \in \mathcal{B}_{i-1}(n,k)$. By Lemma 2.6, we get $\rho_\alpha(\mathcal{H}^\star) > \rho_\alpha(\mathcal{H})$.

By the proofs of Cases (1) and (2), we obtain Claim (1).

If $\mathcal{H} \in \mathcal{B}_4(n,k)$, by Lemma 5.7, we get Lemma 5.8. If $\mathcal{H} \in \mathcal{B}_i(n,k)$ with $i \geq 5$, by Claim (1), there exists a hypergraph $\mathcal{H}^\triangleright \in \mathcal{B}_4(n,k)$ such that $\rho_\alpha(\mathcal{H}^\triangleright) > \rho_\alpha(\mathcal{H})$. Furthermore, by Lemma 5.7, we obtain Lemma 5.8. Thus, Lemma 5.8 holds. □

In Theorem 5.9, we get the hypergraphs with the first and the second largest α-spectral radii among $\mathcal{B}(n,k)$.

Theorem 5.9. *Let $\mathcal{H} \in \mathcal{B}(n,k) \setminus \{\mathcal{A}^{(1)}_{n,k}, \mathcal{B}^{(1)}_{n,k}\}$, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.*

(i). $\rho_\alpha(\mathcal{A}^{(1)}_{n,k}) = \rho_\alpha(\mathcal{B}^{(1)}_{n,k}) > \rho_\alpha(\mathcal{H})$ for $\alpha = 0$.

(ii). $\rho_\alpha(\mathcal{A}^{(1)}_{n,k}) > \rho_\alpha(\mathcal{B}^{(1)}_{n,k}) > \rho_\alpha(\mathcal{H})$ for $0 < \alpha < 1$.

Proof. Let $0 \leq \alpha < 1$, $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$.

By Lemma 4.2, we have $\rho_\alpha(\mathcal{A}^{(1)}_{n,k}) \geq \rho_\alpha(\mathcal{B}^{(1)}_{n,k})$ with the equality if and only if $\alpha = 0$.

If $\mathcal{H} \in \mathcal{B}_2(n,k)$, then $\mathcal{H} = \mathcal{A}_{n,k}(a,b)$. By Lemma 4.3 and Corollary 5.2, we get $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right) \geq \rho_\alpha\left(\mathcal{A}_{n,k}(a,b)\right)$ with the equality if and only if $a \geq b \geq 1$ and $a+b = m-3$. If $\mathcal{H} \in \mathcal{B}_3(n,k)$, then $\mathcal{H}$ is one of the hypergraphs as shown in (20). By Lemma 5.5, we have $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \max\{\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right), \rho_\alpha\left(\mathcal{C}_{n,k}\right), \rho_\alpha\left(\mathcal{F}_{n,k}\right)\} > \rho_\alpha(\mathcal{H})$, where $\mathcal{H} \in \mathcal{B}_3(n,k) \setminus \{\mathcal{B}^{(1)}_{n,k}, \mathcal{C}_{n,k}, \mathcal{F}_{n,k}\}$. If $\mathcal{H} \in \mathcal{B}_i(n,k)$ with $i \geq 4$, then by Lemmas 5.5 and 5.8, we obtain $\rho_\alpha\left(\mathcal{B}^{(1)}_{n,k}\right) > \max\{\rho_\alpha\left(\mathcal{A}^{(2)}_{n,k}\right), \rho_\alpha\left(\mathcal{C}_{n,k}\right), \rho_\alpha\left(\mathcal{F}_{n,k}\right)\} > \rho_\alpha(\mathcal{H})$.

By combining the above proofs, we get Theorem 5.9(i) and (ii). □

Remark 5.10. *Among $\mathcal{B}(n,k)$ with $k = 3$ and $m = \frac{n+1}{k-1} \geq 20$, by the methods similar to those for Theorem 5.9, we obtain the conclusion that the hypergraph with the largest spectral radius is $\mathcal{A}^{(1)}_{n,k}$.*

Remark 5.11. *By the proofs of Theorem 5.9, we get that the hypergraph with the third largest spectral radius among $\mathcal{B}(n,k)$ must be one among $\{\mathcal{A}^{(2)}_{n,k}, \mathcal{C}_{n,k}, \mathcal{F}_{n,k}\}$, where $k \geq 4$ and $m = \frac{n+1}{k-1} \geq 20$. The task will be studied in the future.*

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