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Weak column sufficient tensors and their applications

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Abstract. Column sufficient matrices have wide applications in mathematical analysis, linear complementarity problems, etc., since they contain many important special matrices, such as Hilbert matrices, B-matrices, double B-matrices, H-matrices with nonnegative diagonal entries, P-matrices, M-matrices, and positive (semi-)definite matrices, etc. Column sufficient tensors have recently arisen in connection with the tensor complementarity problem (TCP). By modifying the existing definitions of column sufficient tensors which have some defects for odd order tensors, we propose a formula for the definition of (weak) column sufficient tensor. The proposed column sufficient tensor classes coincide with the existing ones of even orders. We show that all Z-eigenvalues of an even-order symmetric weak column sufficient tensor are nonnegative, and all its H-eigenvalues are nonnegative, regardless of whether the order is even or odd. In contrast to the LCP theory, when the involving tensor in the tensor complementarity problem is a column sufficient tensor, by giving a counterexample, we show that the solution set of the tensor complementarity problem is not necessarily convex. After that several results on tensor complementarity problems are established.

1. Introduction

Column sufficient matrices, first introduced by Cottle et al. [6], is an important type of a special matrix. The class of column sufficient matrices contains many notable matrices as its special cases, such as Hilbert matrices, diagonally dominated matrices with nonnegative diagonal entries, B-matrices, H-matrices with nonnegative diagonal entries, P-matrices, M-matrices, and positive Cauchy matrices. Column sufficiency is useful in justifying the least-index degeneracy resolution scheme in connection with the principal-pivoting method, and it is conjectured that the same is true for Lemke's method (see [4]). The class of column sufficient has recently arisen in connection with the linear complementarity problem (LCP); see [6].

With an emerging interest in multi-linear algebra concentrated on the higher-order tensors, more structured matrices have been generalized to higher-order cases [1, 10, 12, 13, 17–23]. Here the tensor is referred to as a hypermatrix or a multi-way array. In 2018, Chen et al. [2] extended the column sufficient matrix to column sufficient tensor. In the even-order case, it was shown that all H-eigenvalues of a symmetric column sufficient tensor are nonnegative, and all its Z-eigenvalues are nonnegative even in the odd order case.

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As pointed out by Chen et al. [2], most of the spectral properties are only true for the even-order cases and there are no odd-order column sufficient Cauchy tensors according to the definition of Chen et al. [2]. This deficiency is more or less due to the involved non-homogeneous term. In this paper, we will introduce a homogeneous formula for the definition of column sufficient tensor, and discuss their applications in spectral properties and tensor complementarity problems.

Analogous to the matrix case, the class of column sufficient tensors we defined in this paper includes Hilbert tensors [15] diagonally dominated tensors with nonnegative diagonal entries [14], B-tensors [14], double B-tensors [7], quasi-double B-tensors [7], P-tensors [16], strong Hankel tensors [13], M-tensors [10], and positive Cauchy tensors [3].

This paper is organized in the following way. In Section 2, the definitions of weak column sufficient tensors are introduced and several basic properties are characterized. In Section 3, spectral properties of weak column sufficient tensors are studied in the symmetric case, we prove that all H-eigenvalues are nonnegative. For even-order symmetric weak column sufficient tensors, we prove that all its Z-eigenvalues are nonnegative. In Section 4, by constructing a counterexample we show that if $\mathcal{A}$ is a weak column sufficient tensor, need not the solution set SOL($\mathbf{q}, \mathcal{A}$) be convex and show that a relation between any two solutions of TCP($\mathbf{q}, \mathcal{A}$) when $\mathcal{A}$ is column sufficient. Finally, several results on tensor complementarity problems are established.

2. Definitions and basic properties

Notations throughout the paper are introduced as follows. Let $\mathbb{R}$ be the set of all real numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the Euclidean inner product. Denote $[n] := \{1, 2, \dots, n\}$. Vectors are denoted by bold lowercase letters, i.e., $\boldsymbol{x}, \boldsymbol{y}, \dots$, matrices are denoted by capital letters, i.e., $A, B, \dots$, and tensors are written as bold capitals such as $\mathcal{A}, \mathcal{T}, \dots$. Suppose $I \subseteq [n]$. Then $\boldsymbol{x}(I)$ denotes the subvector of $\boldsymbol{x}$ with indices in I. Any vector $\mathbf{x} \in \mathbb{R}^n$ will be considered as column vector/row vector depending upon the context and i-th component of $\mathbf{x}$ denoted by x_i. The zero vector is denoted by $\mathbf{0}$. We write $\mathbf{x} \geq \mathbf{0}$ if each component of $\mathbf{x}$ is nonnegative. Let $\mathbf{x} := (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ and $m \in \mathbb{N}$, we denote $\mathbf{x}^{[m]}$ by $({x_1}^m, {x_2}^m, \dots, {x_n}^m)^T$. Let $\mathbb{R}^{(m,n)}$ be the set of m th-order n-dimensional real tensor. For a tensor $\mathcal{A} = [a_{i_1 i_2 \dots i_m}] \in \mathbb{R}^{(m,n)}$, the entries of the form $a_{ii..i}$ are called diagonal entries of $\mathcal{A}$ and the other entries are called off-diagonal entries. A tensor $\mathcal{A} \in \mathbb{R}^{(m,n)}$ is said to be a diagonal tensor if all its off-diagonal entries are zero. A tensor $\mathcal{A}$ is said to be symmetric if it is invariant under any permutation of their indices. $S(m, n)$ denotes the set of mth-order n-dimensional real symmetric tensors.

Suppose that $\mathcal{A} = (a_{i,1,\dots i_m})$ is a tensor with order m dimension n. Let $I \subseteq [n]$ be an index subset. Then the corresponding principal subtensor $\mathcal{A}(I) = \left(a^I_{i_1 i_2 \cdots i_m}\right)$ is a tensor with order m and dimension $|I|$ such that

$$a^I_{i_1 i_2 \cdots i_m} := a_{i_1 i_2 \cdots i_m}, \quad i_1, i_2, \dots, i_m \in I.$$

Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its i-th component as

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i := \sum_{i_2,\dots,i_m=1}^{n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}$$

for $i \in I_n$. Then $\mathcal{A}\mathbf{x}^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}\mathbf{x}^m := \mathbf{x}^\top \left(\mathcal{A}\mathbf{x}^{m-1}\right) = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m}$$

It is well-known that the linear complementarity problem (LCP) is the first-order optimality conditions of quadratic programming, which has wide applications in applied science and technology such as optimization and physical or economic equilibrium problems. Given a matrix A and a vector $\mathbf{q}$, a linear complementarity problem LCP($\mathbf{q}, A$) is to find a vector $\mathbf{x}$ such that

$$\mathbf{x} \geq \mathbf{0}, \;\; A\mathbf{x} + \mathbf{q} \geq \mathbf{0}, \;\; \langle \mathbf{x}, A\mathbf{x} + \mathbf{q} \rangle = 0$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The corresponding LCP($\mathbf{q}, A$) has a (possibly empty) convex solution set for any vector $\mathbf{q}$ if and only if the matrix A is column sufficient.

The nonlinear complementarity problem has been systematically studied in the mid-1960s and has developed into a very fruitful discipline in the field of mathematical programming, which included a multitude of interesting connections to numerous disciplines and a wide range of important applications in engineering and economics. The tensor complementarity problem is a special structured nonlinear complementarity problem. Corresponding to $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{(m,n)}$ and vector $\mathbf{q} \in \mathbb{R}^n$, The tensor complementarity problem, denoted by TCP ($\mathbf{q}, \mathcal{A}$), is to find $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} \geq \mathbf{0}, \;\; \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{and } \mathbf{x}^\top \left(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}\right) = 0,$$

or to show that no such vector exists. Clearly, the tensor complementarity problem is the first-order optimality condition of the homogeneous polynomial optimization problem, which may be referred to as a direct and natural extension of the linear complementarity problem (for short LCP). The tensor complementarity problem TCP($\mathbf{q}, \mathcal{A}$) is a specially structured nonlinear complementarity problem, and so the TCP($\mathbf{q}, \mathcal{A}$) has its particular and nice properties other than ones of the classical nonlinear complementarity problem.

By definition, a matrix A is a column sufficient if $\boldsymbol{x} \in \mathbb{R}^n$ satisfies

$$x_i \left(A\boldsymbol{x}\right)_i \leqslant 0, \forall i \in [n] \Longrightarrow x_i \left(A\boldsymbol{x}\right)_i = 0, \forall i \in [n].$$

Chen et al. [2] generalized column sufficient matrix to higher order tensors as follows.

Definition 2.1. *An m-th order n-dimensional tensor $\mathcal{A}$ is called a column sufficient tensor (or $\mathcal{A}$ is column sufficient in simple), if $\boldsymbol{x} \in \mathbb{R}^n$ satisfies*

$$x_i \left(\mathcal{A}x^{m-1}\right)_i \leqslant 0, \forall i \in [n] \Longrightarrow x_i \left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n].$$

Unfortunately, it has been shown that there are no odd-order column sufficient Cauchy tensors under Chen, Qi, and Song's definition (see, [2]), and many spectral properties hold only for tensors of even order. To make up for this deficiency, we now propose a modified definition of column sufficient tensor as follows.

Definition 2.2. *An m-th order n-dimensional tensor $\mathcal{A}$ is called a weak column sufficient tensor, if $\boldsymbol{x} \in \mathbb{R}^n$ satisfies*

$$x_i^{m-1} \left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n] \Longrightarrow x_i^{m-1} \left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n] \tag{1}$$

For $X \subseteq \mathbb{R}^n$, if $\boldsymbol{x} \in X$ and (1) holds, then $\mathcal{A}$ is called weak column sufficient in X.

When $m = 2$, these two definitions reduce to the notion of column sufficient matrix [5].
It is clear that our definition is the same as the one of Chen, Qi, and Song in the even-order case. Moreover, the properties about the column sufficient tensor in [2] are certainly valid for weak column sufficient tensors of even orders. Moreover, we will show that most of those properties also hold for weak columns sufficient tensors of odd orders. Recall that a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{(m,n)}$ is called a symmetric tensor if all $a_{i_1 \cdots i_m}$'s are invariant under any permutation of $\{i_1, \ldots, i_m\}$. Chen et al. [2] showed that an even-order Z-tensor $\mathcal{A}$ is column sufficient if and only if $\mathcal{A}$ is column sufficient in $\mathbb{R}^n{}_+$ if and only if the system

$$\begin{cases} \mathcal{A}(I)\boldsymbol{x}^{m-1} \neq 0, \\ \mathcal{A}(I)\boldsymbol{x}^{m-1} \leqslant 0, \\ \boldsymbol{x} > 0, \end{cases} \tag{2}$$

has no solution for any subset $I \subseteq [n]$, where $\boldsymbol{x} \in \mathbb{R}^{|I|}$. Therefore, this equivalence suggests that this properties may be held for weak column sufficient tensors of odd orders.

Definition 2.3. *Let $\mathbb{C}$ be the complex field. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric tensor with order m dimension n. A pair $(\lambda, \boldsymbol{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is called an eigenvalue-eigenvector pair of tensor $\mathcal{A}$, if*

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]}.$$

If the eigenvalue λ and the eigenvector $\boldsymbol{x}$ are real, then λ is called an H-eigenvalue of $\mathcal{A}$ and $\boldsymbol{x}$ is its corresponding H-eigenvector. Moreover, a pair $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times (\mathbb{R}^n \backslash \{0\})$ is called a Z-eigenvalue and Z-eigenvector of tensor $\mathcal{A}$ if they satisfy the following equation system:

$$\begin{cases} \mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x} \\ \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \end{cases}$$

We first prove that for even-order case the two definitions of column sufficiency are equivalent.

Theorem 2.4. *For even m-order n dimentional tensor $\mathcal{A}$, the two definitions 2.1 and 2.2 are equivalent.*

Proof. Assume that $\mathcal{A}$ is a column sufficient tensor with order m dimension n. Suppose that m is even. If possible, let $\boldsymbol{x} \in \mathbb{R}^n$ with $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n]$. If $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$, then $x_i^{m-1} \geqslant 0$. Since m is even, this implies $x_i \geqslant 0$ and so $x_i\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$.
If $x_i^{m-1} \leqslant 0$, then $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \geqslant 0$ and $x_i \leqslant 0$, this implies $x_i\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$. Since $\mathcal{A}$ is column sufficient tensor, it holds that $x_i\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n]$, and so $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n]$. Hence $\mathcal{A}$ is a weak column sufficient tensor.
Now assume that $\mathcal{A}$ is a weak column sufficient tensor. If possible, let $\boldsymbol{x} \in \mathbb{R}^n$ with $x_i\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n]$.
If $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$, then $x_i \geqslant 0$, this implies $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$.
If $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \geqslant 0$, then $x_i \leqslant 0$. Since m is even, this implies $x_i^{m-1} \leqslant 0$ and so $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$.
Since $\mathcal{A}$ is a weak column sufficient tensor, it holds that $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n]$, and so $x_i\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0, \forall i \in [n]$, which implies that $\mathcal{A}$ is a column sufficient tensor. □

The following examples show that there exists a weak column sufficient tensor that is not a column sufficient tensor and vice versa.

Example 2.5. *Let $\mathcal{A} = (a_{i_1 i_2 i_3})$ be a 3-th order 2-dimensional tensor such that*

$$a_{112} = 3, \quad a_{211} = a_{222} = 1,$$

and $a_{i_1 i_2 i_3} = 0$ for the others. Then, for all $\boldsymbol{x} \in \mathbb{R}^2$,

$$x_1^2\left(\mathcal{A}\boldsymbol{x}^2\right)_1 = 3x_1^3 x_2, \quad x_2^2\left(\mathcal{A}\boldsymbol{x}^2\right)_2 = x_2^2\left(x_1^2 + x_2^2\right).$$

Thus, we have

$$x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i \leqslant 0, \forall i \in [2] \Longrightarrow x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i = 0, \forall i \in [2],$$

which means that $\mathcal{A}$ is a weak column sufficient tensor. On the other hand, it is easy to check that $\mathcal{A}$ satisfies

$$x_1\left(\mathcal{A}\boldsymbol{x}_0{}^2\right)_1 = 3x_1^2 x_2 = -3 < 0, \quad x_2\left(\mathcal{A}\boldsymbol{x}_0{}^2\right)_2 = x_2(x_1^2 + x_2^2) = -2 < 0, \quad \boldsymbol{x}_0 = (1, -1) \in \mathbb{R}^2.$$

This implies that $\mathcal{A}$ is not column sufficient tensor.

Example 2.6. *Let $\mathcal{A} = (a_{i_1 i_2 i_3})$ be a 3-th order 2-dimensional tensor such that*

$$a_{112} = a_{122} = -a_{211} = -a_{221} = 1$$

and $a_{i_1 i_2 i_3} = 0$ for the others. Then, for all $\boldsymbol{x} \in \mathbb{R}^2$,

$$x_1\left(\mathcal{A}\boldsymbol{x}^2\right)_1 = x_1 x_2[x_1 + x_2], \quad x_2\left(\mathcal{A}\boldsymbol{x}^2\right)_2 = -x_1 x_2[x_1 + x_2].$$

Thus, we have

$$x_i\left(\mathcal{A}x^2\right)_i \leqslant 0, \forall i \in [2] \Longrightarrow \quad x_1x_2[x_1+x_2] \leqslant 0, -x_1x_2[x_1+x_2] \leqslant 0,$$
$$\Longrightarrow \quad x_1 = 0 \ \ or \ \ x_2 = 0 \ \ or \ \ x_1+x_2=0,$$
$$\Longrightarrow \quad x_i\left(\mathcal{A}x^2\right)_i = 0, \forall i \in [2].$$

which means that $\mathcal{A}$ *is column sufficient tensor. On the other hand, it is easy to check that* $\mathcal{A}$ *satisfies*

$$x_1^2\left(\mathcal{A}x_0{}^2\right)_1 = x_1^2x_2[x_1+x_2] = -4 < 0, \ \ x_2^2\left(\mathcal{A}x_0{}^2\right)_2 = -x_2^2x_1[x_1+x_2] = -2 < 0, \ \ x_0 = (-2,1) \in \mathbb{R}^2.$$

This implies that $\mathcal{A}$ *is not a weak column sufficient tensor.*

To end this section, we present a production between tensors that will be used in the sequel.

Definition 2.7. *Let* $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_2\times\cdots\times n_2}$ *and* $\mathcal{B} \in \mathbb{C}^{n_2\times n_3\times\cdots\times n_{k+1}}$ *be order* $m \geqslant 2$ *and* $k \geqslant 1$ *tensors, respectively. The product* $\mathcal{AB}$ *is the tensor* C *of order* $(m-1)(k-1)+1$ *with entries*

$$c_{i\alpha_1\alpha_2\cdots\alpha_{m-1}} := \sum_{i_2,i_3\ldots,i_m\in[n_2]} a_{ii_2i_3\cdots i_m}b_{i_2\alpha_1}b_{i_3\alpha_2}\cdots b_{i_m\alpha_{m-1}}$$

where $i \in [n_1], \alpha_1,\alpha_2,\ldots,\alpha_{m-1} \in [n_3]\times[n_4]\times\cdots\times[n_{k+1}]$.

Now, we prove the inheritance property of weak column sufficient tensors in the principal subtensor point of view.

Proposition 2.8. *Suppose that* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *is a weak column sufficient tensor with order m dimension n. Then all principal subtensors of* $\mathcal{A}$ *are weak column sufficient tensors.*

Proof. The proof is similar to the proof of Proposition 1 in [2] and so the proof is omitted here. □

Assume $P = \left(p_{ij}\right)$ and $Q = \left(q_{ij}\right)$ are two $n\times n$ matrices. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ be a given tensor with order m dimension n. Then $P\mathcal{A}Q$ is a tensor with order m dimension n such that

$$(P\mathcal{A}Q)_{i_1i_2\cdots i_m} := \sum_{j_1,j_2,\ldots,j_m\in[n]} p_{i_1j_1}a_{j_1j_2\cdots j_m}q_{j_2i_2}q_{j_3i_3}\cdots q_{j_mi_m}$$

Proposition 2.9. *Let* $P = \operatorname{diag}(p_1,p_2,\ldots,p_n)$ *and* $Q = \operatorname{diag}(q_1,q_2,\ldots,q_n)$ *be two diagonal matrices. Suppose that* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *is a weak column sufficient tensor with order m dimension n. If* $p_iq_i^{m-1} > 0$ *for all* $i \in [n]$*, then* $\mathcal{B} = P\mathcal{A}Q$ *is weak column sufficient.*

The following result can be called invariant property under rearrangement of the subscripts.

Proposition 2.10. *Let* $P = \left(p_{ij}\right)$ *be an* $n\times n$ *permutation matrix. Suppose that* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *is a weak column sufficient tensor with order m dimension n. If m is even or* $p_i > 0$ *for all* $i \in [n]$*, Then* $P^{\mathrm{T}}\mathcal{A}P$ *is a weak column sufficient tensor.*

Proof. By Definition 1, assume $\mathcal{B} = P^{\mathrm{T}}\mathcal{A}P$ with entries

$$b_{i_1i_2\cdots i_m} = \sum_{j_1,j_2,\ldots,j_m\in[n]} p_{j_1i_1}a_{j_1j_2\cdots j_m}p_{j_2i_2}p_{j_3i_3}\cdots p_{j_mi_m}$$

Since P is a permutation matrix, only one entry in each row or column is one. Without loss of generality, for any $i \in [n]$, suppose $p_{i'i} = 1$ and $p_{ji} = 0, j \neq i', j \in [n]$. So, it holds that

$$\begin{aligned} b_{i_1i_2\cdots i_m} &= \sum_{j_1,j_2,\ldots,j_m\in[n]} p_{j_1i_1}a_{j_1j_2\cdots j_m}p_{j_2i_2}p_{j_3i_3}\cdots p_{j_mi_m} \\ &= p_{i'_1i_1}a_{i'_1i'_2\cdots i'_m}p_{i'_2i_2}p_{i'_3i_3}\cdots p_{i'_mi_m} \end{aligned}$$

On the other hand, for all $\boldsymbol{x} \in \mathbb{R}^n$ and $i \in [n]$, by above equality,

$$x_i^{m-1}\left(\mathcal{B}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$$

is equivalent to

$$\begin{aligned} x_i^{m-1}\left(\mathcal{B}\boldsymbol{x}^{m-1}\right)_i &= x_i^{m-1}\sum_{i'_2,i'_3,\ldots,i'_m\in[n]} p_{i'i}a_{i'i'_2i'_3\cdots i'_m}p_{i'_2i_2}p_{i'_3i_3}\cdots p_{i'_ii_m}x_{i_2}x_{i_3}\cdots x_{i_m} \\ &= \frac{p_{i'i}^{m-1}}{p_{i'i}^{m-2}}x_i^{m-1}\sum_{i'_2,i'_3,\ldots,i,i'_m\in[n]} a_{i'i'_2i'_3\cdots i'_m}p_{i'_2i_2}p_{i'_3i_3}\cdots p_{i'_ii_m}x_{i_2}x_{i_3}\cdots x_{i_m} \\ &= \frac{y_{i'}^{m-1}}{p_{i'i}^{m-2}}\sum_{i'_2,i'_3,\ldots,i'_m\in[n]} a_{i'i'_2i'_3\ldots i'_m}y_{i'_2}y_{i'_3}\cdots y_{i'_m} \\ &= \frac{y_{i'}^{m-1}}{p_{i'i}^{m-2}}\left(\mathcal{A}\boldsymbol{y}^{m-1}\right)_{i'} \leqslant 0. \end{aligned}$$

where $\boldsymbol{y} = P\boldsymbol{x} \in \mathbb{R}^n$. Since m is even or $p_i > 0$ for all $i \in [n]$, it follows that, for all $i \in [n]$,

$$x_i^{m-1}\left(\mathcal{B}\boldsymbol{x}^{m-1}\right)_i \leqslant 0 \Longleftrightarrow y_{i'}^{m-1}\left(\mathcal{A}\boldsymbol{y}^{m-1}\right)_{i'} \leqslant 0.$$

Combining this with the fact that $\mathcal{A}$ is weak column sufficient, it follows that

$$x_i^{m-1}\left(\mathcal{B}\boldsymbol{x}^{m-1}\right)_i = 0, \quad \forall i \in [n]$$

which, completes the proof. □

A tensor $\mathcal{A} \in T_{m,n}$ is called positive semi-definite if for any vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^m \geq 0$, and is called positive definite if for any nonzero vector $\mathbf{x} \in \mathbb{R}^n, \mathcal{A}\mathbf{x}^m > 0$. Clearly, if m is odd, there are no non-trivial positive semi-definite tensors.

By the definition of positive semi-definite tensors, ([2, Theorem 1]) and Theorem 2.4, we conclude the following theorem.

Theorem 2.11. *Any symmetric positive semidefinite tensor is a weak column sufficient tensor.*

The following example shows that the converse of Theorem 2.11 is not true.

Example 2.12. *Let $\mathcal{A} = (a_{i_1i_2i_3})$ be a 3-th order 2-dimensional tensor such that*

$$a_{121} = a_{112} = a_{211} = a_{222} = 1,$$

and $a_{i_1i_2i_3} = 0$ for the others. Then

$$\mathcal{A}\boldsymbol{x}_0^3 = -4 < 0, \quad \boldsymbol{x}_0 = (1,-1) \in \mathbb{R}^2$$

On the other hand, for all $\boldsymbol{x} \in \mathbb{R}^2$,

$$x_1^2\left(\mathcal{A}\boldsymbol{x}^2\right)_1 = 2x_1^3x_2, \quad x_2^2\left(\mathcal{A}\boldsymbol{x}^2\right)_2 = x_1^2x_2^2 + x_2^4$$

Thus, we have

$$x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i \leqslant 0, \forall i \in [2] \Longrightarrow x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i = 0, \forall i \in [2],$$

which means that $\mathcal{A}$ is a weak column sufficient tensor.

According to Chen, Qi and Song [2], an even-order Z-tensor $\mathcal{A}$ is column sufficient if and only if $\mathcal{A}$ is column sufficient in $\mathbb{R}^n{}_+$ if and only if the system

$$\begin{cases} \mathcal{A}(I)\boldsymbol{x}^{m-1} \neq 0 \\ \mathcal{A}(I)\boldsymbol{x}^{m-1} \leqslant 0 \\ \boldsymbol{x} > 0 \end{cases} \tag{3}$$

has no solution for any subset $I \subseteq [n]$, where $\boldsymbol{x} \in \mathbb{R}^{|I|}$. We will show that These properties also hold for weak column sufficient tensors even in the odd order case.

Theorem 2.13. *Let $\mathcal{A}$ be a Z-tensor with order m dimension n. Then $\mathcal{A}$ is weak column sufficient if and only if $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$*

Proof. The necessary condition is obvious. Conversely, suppose that $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is weak column sufficient in $\mathbb{R}^n_+$ and if possible, let $\boldsymbol{x} \in \mathbb{R}^n$ with $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n]$. Since $\mathcal{A}$ is Z-tensor, this implies

$$\begin{aligned} |x_i|^{m-1}\left(\mathcal{A}|\boldsymbol{x}|^{m-1}\right)_i &= a_{ii\cdots i}\,|x_i|^{2(m-1)} + \sum_{i_2,i_3,\ldots,i_m\in[n],\delta_{ii_2i_3\cdots i_m}=0} a_{ii_2i_3\cdots i_m}\,|x_i|^{m-1}\left|x_{i_2}\right|\left|x_{i_3}\right|\cdots\left|x_{i_m}\right| \\ &\leqslant a_{ii\cdots i}x_i^{2(m-1)} + \sum_{i_2,i_3,\ldots,i_m\in[n],\delta_{ii_2i_3\cdots i_m}=0} a_{ii_2i_3\cdots i_m}x_i^{m-1}x_{i_2}x_{i_3}\cdots x_{i_m} \\ &= x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \\ &\leqslant 0, \quad \forall i \in [n], \end{aligned}$$

where $\delta_{i_1 i_2 \cdots i_m}$ is defined by

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m \\ 0, & \text{otherwise} \end{cases}$$

As $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$, we have

$$|x_i|^{m-1}\left(\mathcal{A}|\boldsymbol{x}|^{m-1}\right)_i \leqslant x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0 \Longrightarrow |x_i|^{m-1}\left(\mathcal{A}|\boldsymbol{x}|^{m-1}\right)_i = x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0$$

for all $i \in [n]$, which yields that $\mathcal{A}$ is a weak column sufficient tensor and the proof is complete. □

Theorem 2.14. *Let $\mathcal{A}$ be a Z-tensor with order m dimension n. Then, $\mathcal{A}$ is a weak column sufficient if and only if the system*

$$\begin{cases} \mathcal{A}(I)\boldsymbol{x}^{m-1} \neq 0, \\ \mathcal{A}(I)\boldsymbol{x}^{m-1} \leqslant 0, \\ \boldsymbol{x} > 0, \end{cases} \tag{4}$$

has no solution for any subset $I \subseteq [n]$, where $\boldsymbol{x} \in \mathbb{R}^{|I|}$.

Proof. For sufficiency, by Theorem 2.11, we only need to prove that $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$. We prove this by contradiction. Assume that $\mathcal{A}$ is not weak column sufficient in $\mathbb{R}^n_+$, which means that there is $\boldsymbol{x} \in \mathbb{R}^n_+$ and at least one index $j \in [n]$ such that

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n], \quad x_j^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_j < 0.$$

Then it holds that

$$I = \{i \in [n] : x_i > 0\} \neq \emptyset.$$

It is not difficult to check that $\boldsymbol{x}(I)$ is a solution of system (4), which is a contradiction. Thus, $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$ and the weak column sufficient condition holds by Theorem 2.11.

On the other hand, supposing that $\mathcal{A}$ is a weak column sufficient tensor, we prove by contradiction that system (4) has no solution. Assume that there is a subset $I \subseteq [n]$ such that system (4) has a solution $\boldsymbol{x} \in \mathbb{R}^{|I|}$, i.e.,

$$\begin{cases} \mathcal{A}(I)\boldsymbol{x}^{m-1} \neq 0, \\ \mathcal{A}(I)\boldsymbol{x}^{m-1} \leqslant 0, \\ \boldsymbol{x} > 0, \end{cases} \tag{5}$$

Define $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $y_i = x_i, i \in I$ and $y_i = 0$ otherwise. By (5), it follows that

$$y_i^{m-1}\left(\mathcal{A}\boldsymbol{y}^{m-1}\right)_i \leqslant 0, \quad \forall i \in [n]$$

and at least one strict inequality holds, which contradicts with the fact that $\mathcal{A}$ is weak column sufficient. □

Cauchy tensor was first studied in [3]. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional Cauchy tensor with generating vector $c = (c_1, c_2, \dots, c_m) \in \mathbb{R}^n$. Then it holds that

$$a_{i_1 i_2 \cdots i_m} := \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \quad \forall i_j \in [n], j \in [m].$$

As pointed out by Chen et al. [2], there is no odd order column sufficient Cauchy tensors. We show that any Cauchy tensor with generating vector $\boldsymbol{c} > 0, \boldsymbol{c} \in \mathbb{R}^n$ is weak column sufficient tensors, regardless of whether the order is even or odd.

Theorem 2.15. *Suppose that $\mathcal{A}$ is an m-th order n-dimensional Cauchy tensor with generating vector $\boldsymbol{c} > 0, \boldsymbol{c} \in \mathbb{R}^n$. Then $\mathcal{A}$ is weak column sufficient.*

Proof. For even m, the result follows from Theorem 1 and Theorem 2.11. Now let m be odd. For any $\boldsymbol{x} \in \mathbb{R}^n$ and $i \in [n]$, we have

$$\begin{aligned} x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i &= x_i^{m-1} \sum_{i_2, i_3, \dots, i_m \in [n]} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{c_i + c_{i_2} + c_{i_3} + \cdots + c_{i_m}} \\ &= x_i^{m-1} \sum_{i_2, i_3, \dots, i_m \in [n]} \int_0^1 t^{c_i + c_{i_2} + c_{i_3} + \cdots + c_{i_m} - 1} x_{i_2} x_{i_3} \cdots x_{i_m} \, \mathrm{d}t \\ &= x_i^{m-1} \int_0^1 \left(\sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j \right)^{m-1} \mathrm{d}t. \end{aligned}$$

So it holds that

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0, \forall i \in [n] \Longleftrightarrow x_i^{m-1} \int_0^1 \left(\sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j \right)^{m-1} \mathrm{d}t \leqslant 0, \forall i \in [n] \tag{6}$$

If $x_i = 0$, then

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = 0$$

Since m is odd, if $x_i \neq 0$, then $x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant 0$ means that

$$\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \int_0^1 \left(\sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j \right)^{m-1} \mathrm{d}t \leqslant 0 \tag{7}$$

Since m is odd

$$\left(\sum_{j\in[n]} t^{c_j+\frac{c_i-1}{m-1}} x_j\right)^{m-1} \geqslant 0, \forall t \in [0,1]$$

and so

$$\int_0^1 \left(\sum_{j\in[n]} t^{c_j+\frac{c_i-1}{m-1}} x_j\right)^{m-1} \mathrm{d}t \geqslant 0$$

Combining this with (7), we have

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = x_i^{m-1}\int_0^1 \left(\sum_{j\in[n]} t^{c_j+\frac{c_i-1}{m-1}} x_j\right)^{m-1} \mathrm{d}t = 0, \quad \forall i \in [n],$$

which implies that $\mathcal{A}$ is weak column sufficient. □

3. Spectral properties of symmetric column sufficient tensors

In this section, spectral properties of weak column sufficient tensors are studied in the symmetric case. Chen et al. [2] showed that for even-order symmetric column sufficient tensors, all its H-eigenvalues are nonnegative, and for general symmetric column sufficient tensors, all its Z-eigenvalues are always nonnegative. Furthermore, it is proved that if all elements of an H-eigenvector or a Z-eigenvector has identical signs, then the corresponding H-eigenvalue or Z-eigenvalue has the same sign. Now we show that all H-eigenvalues of symmetric weak column sufficient tensors are nonnegative, regardless of whether the order is even or odd. For even-order symmetric weak column sufficient tensors, we prove that all its Z-eigenvalues are nonnegative, which implies that an even-order symmetric tensor is a weak column sufficient if and only if it is positive semi-definite. Furthermore, it is proved that if all components of an Z-eigenvector have identical signs, then the corresponding Z-eigenvector has the same sign.

Theorem 3.1. *Let $\mathcal{A}$ be a symmetric weak column sufficient tensor with order m dimension n. Then all H-eigenvalues of $\mathcal{A}$ are nonnegative.*

Proof. Let $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ be an H-eigenpair of tensor $\mathcal{A}$. We know that

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]}.$$

If $\lambda < 0$, then it holds that

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i^{2(m-1)} \leqslant 0, \quad \forall i \in [n].$$

Since $\mathcal{A}$ is weak column sufficient, we have

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i^{2(m-1)} = 0, \quad \forall i \in [n]$$

which implies that $x_1 = x_2 = \cdots = x_n = 0$, and we get a contradiction with $\boldsymbol{x} \neq \mathbf{0}$ Thus, $\lambda \geqslant 0$ and the desired results hold. □

Theorem 3.2. *Let $\mathcal{A}$ be a symmetric weak column sufficient tensor with order m dimension n. Suppose that $\lambda \in \mathbb{R}$ is a Z-eigenvalue of $\mathcal{A}$ with Z-eigenvector $\boldsymbol{x} \in \mathbb{R}^n$. If m is even or $\boldsymbol{x} \geqslant \mathbf{0}$, then $\lambda \geqslant 0$; If $\boldsymbol{x} \leqslant \mathbf{0}$ and m is odd, then $\lambda \leqslant 0$*

Proof. Let $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ be an Z-eigenpair of tensor $\mathcal{A}$. We know that

$$\mathcal{A}x^{m-1} = \lambda \boldsymbol{x}$$

For even m or $\boldsymbol{x} \geqslant \mathbf{0}$, if $\lambda < 0$, then it holds that

$$\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i \leqslant 0, \quad \forall i \in [n]$$

which implies that

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i^m \leqslant 0, \quad \forall i \in [n].$$

Since $\mathcal{A}$ is weak column sufficient, we have

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i^m = 0, \quad \forall i \in [n]$$

which implies that $x_1 = x_2 = \cdots = x_n = 0$, and we get a contradiction with $\boldsymbol{x} \neq \mathbf{0}$ When $\boldsymbol{x} \leqslant \mathbf{0}$ and m is odd, if $\lambda > 0$, then by conditions, we obtain

$$\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i \geqslant 0, \quad \forall i \in [n]$$

which is equivalent to

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \lambda x_i^m \leqslant 0, \quad \forall i \in [n].$$

So, from the fact that $\mathcal{A}$ is weak column sufficient, it follows that $\boldsymbol{x} = \mathbf{0}$, which is a contradiction. Thus, $\lambda \leqslant 0$ and the desired results hold. □

Corollary 3.3. *An even-order real symmetric tensor $\mathcal{A}$ is a weak column sufficient if and only if $\mathcal{A}$ is positive semi-definite.*

Proof. It was proved in [8] that H-eigenvalues and Z-eigenvalues exist for an even-order real symmetric tensor $\mathcal{A}$, and $\mathcal{A}$ is positive semi-definite if and only if all of its H-eigenvalues (Z-eigenvalues) are nonnegative. So, the results follow from Theorems 3.1 and 3.2. □

4. Tensor complementarity problems

In LCP theory (see Theorem 3.5.8 in [5]), if $\mathbf{A}$ is column sufficient matrix, then the solution set to LCP($\mathbf{q}, \mathbf{A}$) is a convex set for all $\mathbf{q} \in \mathbb{R}^n$. When the involving tensor in the tensor complementarity problem is a column sufficient tensor, by giving a counterexample, we show that the solution set of the tensor complementarity problem is not necessarily convex.

Example 4.1. *Let $\mathcal{A} = (a_{i_1 i_2 i_3})$ be a 3-th order 2-dimensional tensor such that*

$$a_{212} = -1, \quad a_{111} = a_{222} = a_{122} = 1,$$

and $a_{i_1 i_2 i_3} = 0$ for the others. Then, for all $\boldsymbol{x} \in \mathbb{R}^2$,

$$x_1^2\left(\mathcal{A}\boldsymbol{x}^2\right)_1 = x_1^2\left(x_1^2 + x_2^2\right), \quad x_2^2\left(\mathcal{A}\boldsymbol{x}^2\right)_2 = x_2^2\left(x_2^2 - x_1 x_2\right).$$

Thus, we have

$$x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i \leqslant 0, \forall i \in [2] \Longrightarrow x_i^2\left(\mathcal{A}\boldsymbol{x}^2\right)_i = 0, \forall i \in [2],$$

which means that $\mathcal{A}$ is a weak column sufficient tensor. On the other hand, it is easy to check that for $q = (-4, 0)$,

$$\mathrm{SOL}(\mathbf{q}, \mathcal{A}) = \left\{(\sqrt{2}, \sqrt{2}), (2, 0)\right\},$$

which is not a convex set.

Definition 4.2. *The k-mode product of a tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ and a vector $\boldsymbol{v} \in \mathbb{R}^n$ is a tensor of order $m-1$. If we denoted by $\mathcal{A} \times_k \boldsymbol{v}$, elementwise,*

$$(\mathcal{A} \times {}_k \boldsymbol{v})_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_N} = \sum_{i_k=1}^{n} a_{i_1,i_2,\ldots,i_m} v_{i_k}.$$

The k-mode product of a tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ and m times a vector $v \in \mathbb{R}^m$ is denoted by $\mathcal{A} \times_1 \boldsymbol{v} \times_2 \boldsymbol{v} \cdots \times_m \boldsymbol{v}$, for brevity by $\mathcal{A}\boldsymbol{v}^m$.

Lemma 4.3. *If $\mathcal{A}$ is a mth order n-dimentional symmetric real tensor, then for every $x, y \in \mathbb{R}^n$,*

$$\mathcal{A}\boldsymbol{x}^{m-1} - \mathcal{A}\boldsymbol{y}^{m-1} = \mathcal{A}(\boldsymbol{x}-\boldsymbol{y})^{m-1}.$$

Proof. For given symmetric tensor $\mathcal{A}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, it results

$$\mathcal{A} \times_k \boldsymbol{x} - \mathcal{A} \times_k \boldsymbol{y} = \mathcal{A} \times_k (\boldsymbol{x}-\boldsymbol{y}).$$

Indeed,

$$\begin{aligned}(\mathcal{A} \times_k \boldsymbol{x})_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} - (\mathcal{A} \times_k \boldsymbol{y})_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} &= \sum_{i_k=1}^{n} a_{i_1,i_2,\ldots,i_m} x_{i_k} - \sum_{i_k=1}^{n} a_{i_1,i_2,\ldots,i_m} y_{i_k} \\ &= \sum_{i_k=1}^{n} a_{i_1,i_2,\ldots,i_m} (x_{i_k} - y_{i_k}) \\ &= (\mathcal{A} \times_k (\boldsymbol{x}-\boldsymbol{y}))_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m}.\end{aligned}$$

Then by computing we have

$$\mathcal{A}\boldsymbol{x}^{m-1} - \mathcal{A}\boldsymbol{y}^{m-1} = \mathcal{A}(\boldsymbol{x}-\boldsymbol{y})^{m-1}.$$

□

Theorem 4.4. *Let $\mathcal{A}$ be a symmetric column sufficient tensor. Assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two solutions of* TCP$(\mathbf{q}, \mathcal{A})$, *then*

$$(\boldsymbol{x})^{\mathrm{T}} \left(\mathcal{A}\boldsymbol{y}^{m-1} + \boldsymbol{q}\right) = (\boldsymbol{y})^{\mathrm{T}} \left(\mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}\right) = 0. \tag{8}$$

Proof. Let $\boldsymbol{w}^1 = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ and $\boldsymbol{w}^2 = \mathcal{A}\boldsymbol{y}^{m-1} + \boldsymbol{q}$. By Lemma 4.3 we have $\boldsymbol{w}^1 - \boldsymbol{w}^2 = \mathcal{A}(\boldsymbol{x}-\boldsymbol{y})^{m-1}$. By the column sufficiency o $\mathcal{A}$ and by the fact that $\boldsymbol{x}$ and $\boldsymbol{y}$ solve the TCP$(\mathbf{q}, \mathcal{A})$, we obtain

$$(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}} \mathcal{A} (\boldsymbol{x}-\boldsymbol{y})^{m-1} = (\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}} \left(\mathcal{A}\boldsymbol{x}^{m-1} - \mathcal{A}\boldsymbol{y}^{m-1}\right) = -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{w}^2 - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{w}^1 \leq 0.$$

Consequently, we must have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{w}^2 = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{w}^1 = 0$, as desired. □

Definition 4.5. *An m-th order n-dimensional tensor $\mathcal{A}$ is called semi-positive if and only if for every $\boldsymbol{x} \in \mathbb{R}^n_+, \boldsymbol{x} \neq \mathbf{0}$, there is an index $i \in [n]$ such that*

$$x_i > 0, \quad \left(\mathcal{A}\boldsymbol{x}^{m-1}\right) > 0.$$

Lemma 4.6. *[20] Tensor $\mathcal{A}$ is semi-positive if and only if* TCP$(\boldsymbol{q}, \mathcal{A})$ *has a unique solution for every $\boldsymbol{q} > \mathbf{0}$.*

Theorem 4.7. *Let $\mathcal{A}$ be a tensor with order m dimension n. If $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$, then $\mathcal{A}$ is semi-positive.*

Proof. The proof technique is similar to the proof technique of ([2], Theorem 13). For completeness, we give the proof here. Let $\boldsymbol{x} \in \mathbb{R}^n_+, \boldsymbol{x} \neq \mathbf{0}$, and $I := \{i \in [n] : x_i > 0\}$. To show that $\mathcal{A}$ is semi-positive, suppose by contradiction that $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i < 0$ for all $i \in I$. Then, we know that

$$x_i^{m-1}\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \begin{cases} = 0, & \forall i \notin I, \\ < 0, & \forall i \in I, \end{cases}$$

which is a contradiction with the fact that $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$. So, for any $\boldsymbol{x} \in \mathbb{R}^n_+, \boldsymbol{x} \neq \mathbf{0}$, there is $x_i > 0$ such that $\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \geqslant 0$ and the desired result holds. □

So, from Lemma 4.6 and Theorem 4.7, we have the following result.

Theorem 4.8. *Let $\mathcal{A}$ be a tensor with order m dimension n. If $\mathcal{A}$ is weak column sufficient in $\mathbb{R}^n_+$, then the* TCP$(\boldsymbol{q}, \mathcal{A})$ *has unique zero solution for every $\boldsymbol{q} > \mathbf{0}$.*

Corollary 4.9. *Suppose that $\mathcal{A}$ is a weak column sufficient tensor with order m dimension n. Then the* TCP$(\boldsymbol{q}, \mathcal{A})$ *has unique zero solution for every $\boldsymbol{q} > \mathbf{0}$.*

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