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Study on CR-submanifolds of Lorentzian para-Kenmotsu manifolds

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Abstract. In this research paper, our investigation focuses on exploring outcomes related to pseudo parallel paracontact CR-submanifolds, considering both Chaki's and Deszcz's definitions. We specifically consider the influence of Levi-Civita connection and semisymmetric metric connection within Lorentzian para-Kenmotsu manifolds.

1. Introduction

The study of the differential geometry of contact CR-submanifolds, which extends the notions of invariant and anti-invariant submanifolds within an almost contact metric manifold, was inaugurated by A. Bejancu [4]. Following this pioneering work, researchers have explored this subject in diverse contexts, as demonstrated in publications like those authored by [10], and [2]. CR-submanifolds refer to a class of submanifolds that are endowed with a distinguished complex structure, which have been extensively studied in differential geometry and complex analysis due to their rich geometric properties and their close connections to various mathematical fields such as algebraic geometry, partial differential equations, and mathematical physics.

The term "CR" stands for Cauchy-Riemann, which reflects the presence of certain geometric conditions that resemble those found in the theory of holomorphic functions. Specifically, CR-submanifolds are equipped with a complex distribution, known as the CR-structure, which captures the complex tangential behavior of these submanifolds.

The study of CR-submanifolds is motivated by their relevance in several areas of mathematics and physics. In complex analysis, they serve as natural objects for the investigation of holomorphic and anti-holomorphic functions on submanifolds, leading to deep connections with the theory of several complex variables. Additionally, CR-submanifolds have found applications in the study of geometric flows, minimal surfaces, and the classification of geometric structures on manifolds. Many significant results are written and published by many authors in relation with CR-submanifolds [7], [1], [6].

By applying the theory of CR-submanifolds to these areas, researchers are able to gain a deeper understanding of the underlying mathematical structures influencing the system's behavior. In 2018, a class of

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Lorentzian manifolds called Lorentzian almost paracontact metric manifolds, known as Lorentzian para-Kenmotsu manifolds, was introduced by [8]. The study of invariant submanifolds of these manifolds was further explored by M. Atceken in 2022, where he provided the necessary and sufficient conditions for a Lorentzian para-Kenmotsu manifold to be totally geodesic [2].

This introduction aims to provide a brief overview of CR-submanifolds, highlighting their importance and relevance in the field of differential geometry. In section 2, we will explore some aspects of CR-submanifolds, including definitions and fundamental properties of CR-submanifolds and Lorentzian para-Kenmotsu manifolds. In section 3, we study result on Chaki-pseudo parallel CR-submanifold of Lorentzian para-Kenmotsu manifold. In section 4, we have established findings regarding Chaki-pseudo parallel CR-submanifolds within Lorentzian para-Kenmotsu manifolds concerning a semisymmetric metric connection. Moreover, we will delve into the associated geometric structures and the intricate interplay between totally geodesic and Lorentzian para-Kenmotsu manifolds in the context of CR-submanifolds. By delving into the fascinating world of CR-submanifolds, we hope to gain insights into the deep connections between complex analysis, differential geometry, and various other mathematical disciplines.

2. Preliminaries

Let M be the odd-dimensional Lorentzian metric manifold. To say M is Lorentzian almost contact manifold it should have the structure (ϕ, ξ, η, g), where ϕ, ξ, η, g denotes the $(1, 1)$ tensor, vector field, 1-form and Lorentz metric respectively satisfying

$$\phi^2\Omega_1 = \Omega_1 + \eta(\Omega_1)\xi, \quad \phi\xi = 0, \tag{1}$$

$$g(\phi\Omega_1, \phi\Omega_2) = g(\Omega_1, \Omega_2) + \eta(\Omega_1)\eta(\Omega_2), \tag{2}$$

$$\eta(\xi) = -1, \tag{3}$$

$$\eta(\Omega_1) = g(\Omega_1, \xi). \tag{4}$$

A Lorentzian almost paracontact manifold $\overset{\star}{M}{}^n(\phi, \xi, \eta, g)$ is treated as a Lorentzian para-Kenmotsu manifold($\mathcal{LPKM}$) if the below defined conditions is satisfied for all Ω_1 and Ω_2 in the set of differentiable vector fields $\Gamma(T\overset{\star}{M}{}^n)$. Here, $\bar{\nabla}$ denotes the Levi-Civita connection($\mathcal{LCC}$):

$$(\bar{\nabla}_{\Omega_1}\phi)\Omega_2 = -g(\phi\Omega_1, \Omega_2)\xi - \eta(\Omega_2)\phi\Omega_1 \tag{5}$$

Moreover, when ξ represents the Killing vector field, the contact structure is termed a K-contact (or para-contact) structure. In this case, the following relationship holds:

$$\nabla_{\Omega_1}\xi = \phi\Omega_1 \tag{6}$$

In condition to the above condition for the $\mathcal{LPKM}$ $\overset{\star}{M}{}^n(\phi, \xi, \eta, g)$, also possesses these conditions,

$$\bar{\nabla}_{\Omega_1}\xi = -\phi^2\Omega_1 = -\Omega_1 - \eta(\Omega_1)\xi, \tag{7}$$

$$(\bar{\nabla}_{\Omega_1}\eta)\Omega_2 = -g(\Omega_1, \Omega_2) - \eta(\Omega_1)\eta(\Omega_2) \tag{8}$$

Let $\bar{R}$ be Riemannian curvature tensor and S be Ricci tensor of $\mathcal{LPKM}$ $\overset{\star}{M}{}^n(\phi, \xi, \eta, g)$, then we have

$$R(\Omega_1, \Omega_2)\xi = \eta(\Omega_2)\Omega_1 - \eta(\Omega_1)\Omega_2, \tag{9}$$

$$\bar{R}(\xi, \Omega_1)\Omega_2 = g(\Omega_1, \Omega_2)\xi - \eta(\Omega_2)\Omega_1, \tag{10}$$

$$S(\xi, \Omega_1) = (n-1)\eta(\Omega_1) \tag{11}$$

For the immersed submanifold M of $\mathcal{LPKM}$ $\overset{\star}{M}{}^n$, we denote the tangent and normal subspace by $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$. Then the Gauss and Weigarten formulas are as follows,

$$\bar{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2) \tag{12}$$

$$\bar{\nabla}_{\Omega_1}V = -A_V\Omega_1 + \nabla^{\perp}_{\Omega_1}V \tag{13}$$

for all $\Omega_1, \Omega_2 \in \Gamma(TM)$. Let $V \in \Gamma(T^{\perp}M)$, where ∇, and $\nabla^{\perp}$ are the connections on M. π and A are the second fundamental form($\mathcal{SFF}$) and shape operator of M, respectively. They are interconnected by the following relation,

$$g(A_V\Omega_1, \Omega_2) = g(\pi(\Omega_1, \Omega_2), V) \tag{14}$$

for any $\Omega_1, \Omega_2 \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where g denotes the Riemannian metric on $\overset{\star}{M}{}^n$ and also on M.

For any submanifold M of a Riemannian manifold, the Gauss equation is expressed as follows,

$$\begin{aligned}\bar{R}(\Omega_1, \Omega_2)\Omega_3 = R(\Omega_1, \Omega_2)\Omega_3 + A_{\pi(\Omega_1,\Omega_2)}\Omega_2 - A_{\pi(\Omega_2,\Omega_3)}\Omega_1 \\ +(\bar{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) - (\bar{\nabla}_{\Omega_2}\pi)(\Omega_1, \Omega_3)\end{aligned} \tag{15}$$

The covariant derivative of π is expressed as follows:

$$(\bar{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) = \nabla^{\perp}_{\Omega_1}\pi(\Omega_2, \Omega_3) - \pi(\nabla_{\Omega_1}\Omega_2, \Omega_3) - \pi(\Omega_2, \nabla_{\Omega_1}\Omega_3) \tag{16}$$

for all $\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM)$, here $\bar{R}$ and R denotes the Riemannian curvature tensors of $\bar{M}$ and M respectively. If $\bar{\nabla}\pi = 0$, then the submanifold M is considered to have its $\mathcal{SFF}$ [9].

The normal part $(\bar{R}(\Omega_1, \Omega_2)\Omega_3)^{\perp}$ of $(\bar{R}(\Omega_1, \Omega_2)\Omega_3)$ from (15) is given by

$$(\bar{R}(\Omega_1, \Omega_2)\Omega_3)^{\perp} = (\bar{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) - (\bar{\nabla}_{\Omega_2}\pi)(\Omega_1, \Omega_3). \tag{17}$$

This equation is commonly referred to as the Codazzi equation. In particular, if $(\bar{R}(\Omega_1, \Omega_2)\Omega_3)^{\perp} = 0$, then M is referred to as a curvature-invariant submanifold of $\overset{\star}{M}$.

Alternatively, as M is tangent to ξ, we have

$$A_V\xi = \pi(\Omega_1, \xi) = 0. \tag{18}$$

By using above equation (18), we have from (7) and (12) i.e.,

$$\nabla_{\Omega_1}\xi = \bar{\nabla}_{\Omega_1}\xi = \Omega_1 - \eta(\Omega_1)\xi. \tag{19}$$

In view of (18), we have from (15) and (10) that,

$$R(\Omega_1, \Omega_2)\xi = \bar{R}(\Omega_1, \Omega_2)\xi = \eta(\Omega_1)\Omega_2 - \eta(\Omega_2)\Omega_1. \tag{20}$$

In almost contact metric manifolds, the categorization of invariant and anti-innvariant submanifolds relies on the properties of almost contact metric structure ϕ. A submanifolds M in an almost contact metric manifold is termed invariant if the structure vector field ξ is tangent to M at every point and $\phi\Omega_1$ is tangent to M for any vector field Ω_1 that is tangent to M. This condition can be represented as $\phi(TM) \subset TM$ at every point in M. Now, for invariant submanifolds of a $\mathcal{LPKM}$, the manifold is said to be totally geodesic($\mathcal{T}G$) if π is identically zero (as stated in reference [5]):

$$\pi(\Omega_1, \xi) = 0. \tag{21}$$

A linear connection on a $\mathcal{LPKM}$ $\overset{\star}{M}{}^n$ is termed a semisymmetric connection if its torsion tensor τ of the connection $\bar{\bar{\nabla}}$ is structured in the following manner:

$$\tau(\Omega_1, \Omega_2) = \bar{\bar{\nabla}}_{\Omega_1}\Omega_2 - \bar{\bar{\nabla}}_{\Omega_2}\Omega_1 - [\Omega_1, \Omega_2] \tag{22}$$

satisfies,

$$\tau(\Omega_1, \Omega_2) = \eta(\Omega_2)\Omega_1 - \eta(\Omega_1)\Omega_2, \tag{23}$$

wher η is a 1-form. Moreover, if the semisymmetrc connection $\bar{\bar{\nabla}}$ satisfies the condition

$$(\bar{\bar{\nabla}}_{\Omega_1} g)(\Omega_1, \Omega_2) = 0. \tag{24}$$

for all $\Omega_1, \Omega_2, \Omega_3 \in \chi(\overset{\star}{M}^n)$, where $\chi(\overset{\star}{M}^n)$ represents the lie algebra of vector fields on the manifold $\overset{\star}{M}^n$, then $\bar{\bar{\nabla}}$ is said to be a semisymmetric metric connection ($\mathcal{S}SMC$).

Let $\overset{\star}{M}^n$ be an n-dimensional $\mathcal{LPKM}$ and $\mathcal{SSMC}$ $\bar{\bar{\nabla}}$ in a $\mathcal{LPKM}$ then we have

$$\bar{\bar{\nabla}}_{\Omega_1}\Omega_2 = \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi. \tag{25}$$

Definition 2.1. *A submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\bar{\nabla}$ is called pseudo parallel(PP) if its $\mathcal{SFF}$ π satisfies,*

$$(\bar{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2) \tag{26}$$

for all $\Omega_1, \Omega_2, \Omega_3$ on M, where α is a nowhere vanishing 1-form.

Specifically, if α is a non-vanishing 1-form, then if $\alpha(\Omega_1) = 0$, π is denoted as parallel, and M is termed a parallel submanifold of $\overset{\star}{M}^n$. We proceed to establish the following:

In particular, if $\alpha(\Omega_1) = 0$ then π is said to be parallel and M is said to be parallel submanifold of $\overset{\star}{M}^n$. We now prove the following:

3. Chaki-pseudo parallel CR-submanifold of Lorentzian para-Kenmotsu manifold($\mathcal{LPKM}$)

Theorem 3.1. *Let M the a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$. Then M is $\mathcal{T}G$ if and only if M is Chaki-pseudo parallel with $\alpha(\xi) \neq 1$.*

Proof. *Let us take M as Chaki-pseudo parallel CR-submanifold of $\overset{\star}{M}^n$. Then by considering equations* (16) *and* (26)

$$\nabla^{\perp}_{\Omega_1}(\pi(\Omega_2, \Omega_3)) - \pi(\nabla_{\Omega_1}\Omega_2, \Omega_3) - \pi(\Omega_2, \nabla_{\Omega_1}\Omega_3) \tag{27}$$
$$= 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2), \tag{28}$$

Substituting $\Omega_3 = \xi$ in above eqaution and using (21) *we compute*

$$-\pi(\Omega_2, \nabla_{\Omega_1}\xi) = \alpha(\xi)\pi(\Omega_1, \Omega_2). \tag{29}$$

In view of (7), (15) *and* (29) *we get*

$$[1 - \alpha(\xi)]\pi(\Omega_1, \Omega_2) = 0. \tag{30}$$

Here we can see that $\pi(\Omega_1, \Omega_2) = 0$ for all Ω_1, Ω_2 on M as $\alpha(\xi) \neq 1$. Therfore M is $\mathcal{T}G$ submanifold. Converse part is also trivial. Hence the proof. □

Corollary 3.2. *[3] Let M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$. Then M is $\mathcal{T}G$ if and only if M is parallel.*

By taking the reference of the definition of Ricci pseudosymmetric manifold with respect to the Deszcz, we can define the following:

Definition 3.3. *A submanifold M of $\mathcal{LPKM}\ \overset{\star}{M}^n$ is said to be pseudo parallel($\mathcal{PP}$) with respect to Deszcz if its $\mathcal{SFF}$ π satisfies*

$$\begin{aligned}\bar{R}(\Omega_1, \Omega_2).\pi &= (\bar{\nabla}_{\Omega_1}\bar{\nabla}_{\Omega_2} - \bar{\nabla}_{\Omega_2}\bar{\nabla}_{\Omega_1} - \bar{\nabla}_{[\Omega_1,\Omega_2]})\pi \\ &= L_1 Q(g, \pi)\end{aligned} \tag{31}$$

for all vector fields Ω_1, Ω_2 tangent to M, where $\bar{R}$ is the curvature tensor of $\bar{M}$. If $L_1 = 0$ then M is said to be semiparallel($\mathcal{SP}$).

We now prove the following:

Theorem 3.4. *Let M be a CR-submanifold of a $\mathcal{LPKM}\ \overset{\star}{M}^n$. Then M is $\mathcal{T}G$ if and only if M is $\mathcal{PP}$ with respect to Deszcz ($L_1 \neq -1$).*

Proof. *Let M be a contact CR-submanifold of a $\mathcal{LPKM}\ \overset{\star}{M}^n$.*
If M is $\mathcal{PP}$ with respect to the Deszcz. Then from the relation (31) *we have*

$$(\bar{R}(\Omega_1, \Omega_2).\pi)(\Omega_3, \Omega_4) = L_1 Q(g, \pi)(\Omega_3, \Omega_4; \Omega_1, \Omega_2). \tag{32}$$

By using the formula of tensor algebra we have that

$$(\bar{R}(\Omega_1, \Omega_2).\pi)(\Omega_3, \Omega_4) = R^{\perp}(\Omega_1, \Omega_2)\pi(\Omega_3, \Omega_4) - \pi(R(\Omega_1, \Omega_2)\Omega_3, \Omega_4) - \pi(\Omega_3, R(\Omega_1, \Omega_2)\Omega_4) \tag{33}$$

for all vector fields $\Omega_1, \Omega_2, \Omega_3$ and Ω_4, where

$$R^{\perp}(\Omega_1, \Omega_2) = [\nabla^{\perp}_{\Omega_1}, \nabla^{\perp}_{\Omega_2}] - \nabla^{\perp}_{[\Omega_1,\Omega_2]}. \tag{34}$$

Taking the account of equation (1.3), we have

$$Q(g, \pi)(\Omega_3, \Omega_4; \Omega_1, \Omega_2) = g(\Omega_2, \Omega_3)\pi(\Omega_1, \Omega_4) - g(\Omega_1, \Omega_3)\pi(\Omega_2, \Omega_4) \tag{35}$$
$$+ g(\Omega_2, \Omega_4)\pi(\Omega_1, \Omega_4) - g(\Omega_1, \Omega_4)\pi(\Omega_2, \Omega_4). \tag{36}$$

In view of (33) *and* (35) *and also using* (32), *we have that*

$$\begin{aligned}&R^{\perp}(\Omega_1, \Omega_2)\pi(\Omega_3, \Omega_4) - \pi(R(\Omega_1, \Omega_2)\Omega_3, \Omega_4) - \pi(\Omega_3, R(\Omega_1, \Omega_2)\Omega_4) \\ &= L_1[g(\Omega_2, \Omega_3)\pi(\Omega_1, \Omega_4) - g(\Omega_1, \Omega_3)\pi(\Omega_2, \Omega_4) + g(\Omega_2, \Omega_4)\pi(\Omega_1, \Omega_3) - g(\Omega_1, \Omega_4)\pi(\Omega_2, \Omega_3)].\end{aligned} \tag{37}$$

Replacing ξ in Ω_1 and Ω_4 place and using (12) *we get,*

$$\pi(\Omega_3, R(\xi, \Omega_2)\xi) = L_1\pi(\Omega_2, \Omega_3) \tag{38}$$

Putting value of equation (20) *in* (38) *and using* (18) *we get $(L_1 + 1)\pi(\Omega_2, \Omega_3) = 0$, which implies that $\pi(\Omega_2, \Omega_3) = 0$ for all Ω_2, Ω_3 on M, which implies that M is $\mathcal{T}G$, since $L_1 \neq -1$. The converse part is also holds trivial. Hence the proof.* □

Corollary 3.5. *Let M be a CR-submanifold of a $\mathcal{LPKM}\ \overset{\star}{M}^n$. Then M to be $\mathcal{T}G$ if and only if M is $\mathcal{SP}$.*

From Corollary (3.2),(3.5) and Theorem (3.1), (3.4) we can state the following:

Theorem 3.6. *Let M be a CR-submanifold of a $\mathcal{LPKM}\ \overset{\star}{M}^n$. Then the following statements are equivalent:*

1. *M is parallel,*
2. *M is $\mathcal{T}G$,*
3. *M is $\mathcal{SP}$,*
4. *M is $\mathcal{PP}$ with respect to Chaki with $\alpha(\xi) \neq 1$.*
5. *M is $\mathcal{PP}$ with respect to Deszcz with $L_1 \neq -1$.*

4. Chaki-pseudo parallel CR- submanifold of Lorentzian para-Kenmotsu manifold($\mathcal{LPKM}$) with respect to semisymmetric metric connection($\mathcal{SSMC}$)

Let us consider M as a CR-submanifod of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{LCC}$ $\bar{\nabla}$ and $\mathcal{SSMC}$ $\bar{\tilde{\nabla}}$. Let ∇ be the induced connection on M from the connection $\bar{\nabla}$ and $\tilde{\nabla}$ be the induced connection on M from the connection $\bar{\tilde{\nabla}}$. The fundamental form with respect to $\mathcal{LCC}$ and $\mathcal{SSMC}$ are denoted by π and $\tilde{\pi}$ respectivey, then we have

$$\bar{\tilde{\nabla}}_{\Omega_1}\Omega_2 = \tilde{\nabla}_{\Omega_1}\Omega_2 + \tilde{\pi}(\Omega_1, \Omega_2). \tag{39}$$

By using the equation (12) and (25), we have from (39) that

$$\begin{aligned}\tilde{\nabla}_{\Omega_1}\Omega_2 + \tilde{\pi}(\Omega_1, \Omega_2) =& \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi \\ =& \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2) + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi.\end{aligned} \tag{40}$$

Here we have $\Omega_1, \xi \in TM$, we equate the tangential and normal components of (39) we have,

$$\tilde{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi, \tag{41}$$

and

$$\tilde{\pi}(\Omega_1, \Omega_2) = \pi(\Omega_1, \Omega_2) \tag{42}$$

It can be observed that the $\mathcal{SFF}$, as defined by the $\mathcal{LCC}$ and the $\mathcal{SSMC}$, are identical.

As a result the following can be inferred.

Theorem 4.1. *If M is a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to a $\mathcal{SSMC}$, then*

1. *M admits $\mathcal{SSMC}$.*
2. *The $\mathcal{SFF}$ with respect to Riemannian connection and $\mathcal{SSMC}$ are equal.*

If M be a CR-submanifold of a $\mathcal{LPKM}$ is peudo parallel in the sense of Chaki with respect to $\mathcal{SSMC}$. Then we have

$$(\tilde{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2), \tag{43}$$

for all $\Omega_1, \Omega_2, \Omega_3$ on M.

In view of (41) and (18) we have from (43)

$$\begin{aligned}&(\nabla_{\Omega_1}\pi)(\Omega_2, \Omega_3) + g(\pi(\Omega_2, \Omega_3), \xi) - g(\Omega_1, \pi(\Omega_2, \Omega_3))\xi \\ &- \eta(\Omega_2)\pi(\Omega_1, \Omega_3) - \eta(\Omega_3)\pi(\Omega_1, \Omega_2) \\ &= 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2),\end{aligned} \tag{44}$$

which implies that,

$$\begin{aligned}&\nabla^{\perp}_{\Omega_1}\pi(\Omega_2, \Omega_3) - \pi(\nabla_{\Omega_1}\Omega_2, \Omega_3) - \pi(\Omega_2, \nabla_{\Omega_1}\Omega_3) \\ &+ g(\pi(\Omega_2, \Omega_3), \xi) - g(\Omega_1, \pi(\Omega_2, \Omega_3))\xi \\ &- \eta(\Omega_2)\pi(\Omega_1, \Omega_3) - \eta(\Omega_3)\pi(\Omega_1, \Omega_2) \\ &= 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2).\end{aligned} \tag{45}$$

Substituting $\Omega_3 = \xi$ in (45) and utilizing the equation (7), we have

$$\alpha(\xi)\pi(\Omega_1, \Omega_2) = 0 \tag{46}$$

which implies $\pi(\Omega_1, \Omega_2) = 0$ provided $\alpha(\xi) \neq 0$. Now we state the following:

Theorem 4.2. *Let M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. Then M is $\mathcal{TG}$ if and only if M is Chaki $\mathcal{PP}$ with respect to $\mathcal{SSMC}$, provided $\alpha(\xi) \neq 0$.*

Corollary 4.3. *Let M be a CR- submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. Then M is $\mathcal{TG}$ if and only if M is parallel with respect to $\mathcal{SSMC}$.*

Definition 4.4. *A submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ which is associated with $\mathcal{SSMC}$ is said to be $\mathcal{PP}$ in the sense of Deszcz with respect to $\mathcal{SSMC}$ if*

$$\bar{\tilde{R}}(\Omega_1, \Omega_2).\tilde{\pi} = L_1 Q(g, \tilde{\pi}) \tag{47}$$

this condition holds for all vector fields Ω_1, Ω_2 tangent to M, where the curvature tensor of $\overset{\star}{M}^n$ is denoted by $\bar{\tilde{R}}$. We can say that M is $\mathcal{SP}$ with respect to $\mathcal{SSMC}$ if $L_1 = 0$.

Now we prove following theorem:

Theorem 4.5. *Let M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. Then M is $\mathcal{TG}$ if and only if M is Deszcz $\mathcal{PP}$ with respect to $\mathcal{SSMC}$, provided $L_1 \neq -1$.*

Proof. *Let M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. Suppose that M is $\mathcal{PP}$ in the sense of Deszcz with respect to $\mathcal{SSMC}$. Then we have from (47) that*

$$\bar{\tilde{R}}(\Omega_1, \Omega_2).\pi = L_1 Q(g, \pi) \tag{48}$$

which implies that,

$$\begin{aligned} &\bar{R}^{\perp}(\Omega_1, \Omega_2)\pi(\Omega_3, \Omega_4) - \pi(\tilde{R}(\Omega_1, \Omega_2)\Omega_3, \Omega_4) - \pi(\Omega_3, \bar{R}(\Omega_1, \Omega_2)\Omega_4) \\ &= L_1[g(\Omega_2, \Omega_3)\pi(\Omega_1, \Omega_4) - g(\Omega_1, \Omega_3)\pi(\Omega_2, \Omega_4) + g(\Omega_2, \Omega_4)\pi(\Omega_1, \Omega_3) \quad (49) \\ &- g(\Omega_1, \Omega_4)\pi(\Omega_2, \Omega_3)]. \quad (50) \end{aligned}$$

Substitute $\Omega_1 = \Omega_4 = \xi$ in (49) *and also we consider the equation* (7) *we obtain*

$$(L_1 + 1)\pi(\Omega_2, \Omega_3) = 0, \tag{51}$$

then for all Ω_1, Ω_2 on M $\pi(\Omega_2, \Omega_3) = 0$. Therefore M is $\mathcal{TG}$, since $L_1 \neq -1$. Converse part also holds trivial. Hence the result. □

Corollary 4.6. *Let M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. Then M is $\mathcal{TG}$ if and only if M is $\mathcal{SP}$ with respect to $\mathcal{SSMC}$.*

From Corollary (4.3), (4.6) *and theorem* (4.1), (4.5), *we can state the following:*

Theorem 4.7. *If M be a CR-submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\mathcal{SSMC}$. The following statements are equivalent:*

1. *M is $\mathcal{TG}$,*
2. *M is parallel with respect to $\mathcal{SSMC}$,*
3. *M is $\mathcal{SP}$ with respect to $\mathcal{SSMC}$,*
4. *M is $\mathcal{PP}$ in the sense of Chaki with respect to $\mathcal{SSMC}$ with $\alpha(\xi) \neq 0$,*
5. *M is $\mathcal{PP}$ in the sense of Deszcz with respect to $\mathcal{SSMC}$ with $L_1 \neq -1$.*

5. Conclusion

In this paper, we investigate pseudo parallel CR-submanifolds of $\mathcal{LPKM}$, focusing on the notions of pseudo parallelism as defined by Chaki and Deszcz. It is noteworthy that pseudo Ricci symmetric manifolds or $\mathcal{PP}$ manifolds according to Chaki's definition are different from Ricci pseudosymmetric manifolds or $\mathcal{PP}$ manifolds as per Deszcz's characterization. Nevertheless, we prove the equivalence of $\mathcal{PP}$ CR-submanifolds, characterized both by Chaki and Deszcz, within $\mathcal{LPKM}$ under specific conditions. Furthermore, we establish the equivalence of $\mathcal{PP}$ contact CR-submanifolds, considering both Chaki's and Deszcz's definitions, with respect to a $\mathcal{SSMC}$ within $\mathcal{LPKM}$, subject to specific conditions.

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