Filomat 39:9 (2025), 2913–2927
https://doi.org/10.2298/FIL2509913L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Batalin-Vilkovisky structures on the cohomologies of tensor Poisson algebras

J. Luo[a], S.-Q. Wang[b,*]

[a]*Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China*
[b]*School of Mathematics, East China University of Science and Technology, Shanghai 200237, China*

Abstract. It has been proved that the Poisson cohomology ring of a Poisson algebra has a Batalin-Vilkovisky algebra structure iff the Poisson algebra is pseudo-unimodular. In this paper, it is proved that the tensor algebra of two pseudo-unimodular Poisson algebras is also pseudo-unimodular, thus its Poisson cohomology ring is still a Batalin-Vilkovisky algebra. Furthermore, This Batalin-Vilkovisky algebra is isomorphic to the tensor Batalin-Vilkovisky algebra of the respective Poisson cohomology rings of the two pseudo-unimodular Poisson algebras.

1. Introduction

Poisson algebras and their (co)homology theory play an important role in the study of their deformation quantization algebras, since Poisson algebras and their deformation quantization algebras often share many similar homological properties. For example, Dolgushev proved that the Van den Bergh duality holds for the deformation quantizations of unimodular Poisson algebras [8]. And the Hochschild homology of some Calabi-Yau algebras have been calculated while they are viewed as deformations of unimodular polynomial Poisson algebras [4, 24, 26, 27, 29]. The cohomologies and deformations of tensor Poisson algebras are investigated in [7].

Unimodular Poisson structures attract much attention since they have nice properties, such as Poincaré duality and the Batalin-Vilkovisky (BV for short) structures on their Poisson cohomologies [5, 33]. In fact, for any smooth Poisson algebra with trivial canonical bundle or Frobenius Poisson algebra, twisted Poincaré duality always holds by twisting the Poisson module structure in a canonical way, which is constructed from the modular derivation[12, 15, 19, 22, 34].

On the other hand, Gerstenhaber structures and BV structures on cohomologies have been widely studied since they appear in the research of BV formalism and play an important role in quantum field theory and string theory [25]. The BV structures on the Hochschild (co)homology of noncommutative algebras have been considered by many researchers, such as [10, 13, 14, 28]. In [17], Le and Zhou considered the tensor product of two Gerstenhaber algebras, which is still a Gerstenhaber algebras, and proved that as

2020 *Mathematics Subject Classification*. Primary 17B63; Secondary 16E40.
Keywords. Poisson algebra, tensor product, Batalin-Vilkovisky algebra, smooth algebra, Frobenius algebra.
Received: 04 June 2024; Accepted: 08 January 2025
Communicated by Dijana Mosić
* Corresponding author: S.-Q. Wang
Email addresses: luojuan@shnu.edu.cn (J. Luo), sqwang@ecust.edu.cn (S.-Q. Wang)
ORCID iDs: https://orcid.org/0009-0008-6813-6074 (J. Luo), https://orcid.org/0009-0003-6560-5852 (S.-Q. Wang)

Gerstenhaber algebras, the Hochschild cohomology ring of the tensor product of two algebras is isomorphic to the tensor product of the respective Hochschild cohomology rings of these two algebras when at least one of which is finite dimensional. Zhu generalized this to the Poisson framework. He proved that the Poisson cohomology ring of the tensor product of two Poisson algebras is isomorphic to the tensor product of the respective Poisson cohomology rings of these two Poisson algebras as Gerstenhaber algebras [35]. Since BV algebras are special Gerstenhaber algebras whose Gerstenhaber brackets can be defined by BV operators, we consider the BV structures on the Poisson cohomology of the tensor Poisson algebra in this paper, while the unimodular polynomial Poisson algebras case has been investigated in [6].

As a generalization of unimodular Poisson structures, a notion of pseudo-unimodular Poisson structure is defined for smooth algebras with trivial canonical bundles [21] and Frobenius algebras [20]. It is proved that the Poisson cohomology ring of a Poisson algebra can be endowed with a BV algebra structure inherited from some one of its Poisson cochain complex if and only if the Poisson algebra is pseudo-unimodular. The mail result in this paper is that the tensor algebra of two pseudo-unimodular Poisson algebras is also pseudo-unimodular and its Poisson cohomology ring is isomorphic to the tensor product of the respective Poisson cohomology rings of these two Poisson algebras as BV algebras.

This paper is organized as follows. In Section 1, we recall some preliminary definitions and results mainly on Poisson algebras and cohomology rings, pseudo-unimodular Poisson structures, Gerstenhaber algebras and BV algebras, etc. In Section 2, we prove that the tensor algebra of two pseudo-unimodular Poisson algebras is still pseudo-unimodular. Then we study the BV structure on the Poisson cohomology of the tensor algebra of two pseudo-unimodular smooth Poisson algebras with trivial canonical bundles, and prove that it is isomorphic to the tensor BV algebra of the respective Poisson cohomology rings. In Section 3, we also consider the BV algebra isomorphism for Frobenius Poisson algebras.

2. Preliminaries

In this section, we collect some necessary materials on Poisson algebras, Poisson cohomology, Gerstenhaber algebras and Batalin-Vilkovisky algebras. Let $\Bbbk$ be a field. All vector spaces and algebras are over $\Bbbk$. We refer to [16] as the basic reference.

2.1. Poisson algebras and Poisson cohomology

Definition 2.1. *[18, 32] A commutative $\Bbbk$-algebra R equipped with a bilinear map $\pi = \{-,-\} : R \times R \to R$ is called a* **Poisson algebra***, denoted by (R, π), if*

1. *the underline $\Bbbk$-vector space R together with $\{-,-\} : R \times R \to R$ is a $\Bbbk$-Lie algebra;*
2. *$\{-,-\} : R \times R \to R$ is a derivation in each argument with respect to the multiplication of R.*

For any $p \in \mathbb{N}$, let $\mathfrak{X}^p(R)$ be the set of all skew-symmetric $\Bbbk$-linear maps from $R^{\otimes p}$ to R which are derivations in each argument, that is,

$$\mathfrak{X}^p(R) = \{F \in \mathrm{Hom}_{\Bbbk}(\wedge^p R, R) \mid F \text{ is a derivation in each argument}\}.$$

Obviously, $\mathfrak{X}^1(R) = \mathrm{Der}_{\Bbbk}(R)$ is the set of $\Bbbk$-linear derivations of R. Let $\Omega^1(R)$ be the canonical module of Kähler differentials with $\mathrm{d} : R \to \Omega^1(R)$ the classical de Rham differential, and $\Omega^p(R) = \wedge_R^p \Omega^1(R)$ be its p-th wedge product. Then for any $p \in \mathbb{N}$,

$$\mathfrak{X}^p(R) \cong \mathrm{Hom}_R(\Omega^p(R), R). \tag{2.1}$$

Set $\mathfrak{X}^*(R) = \oplus_{p \in \mathbb{N}} \mathfrak{X}^p(R)$ and $\Omega^*(R) = \oplus_{p \in \mathbb{N}} \Omega^p(R)$. There is a canonical cochain complex

$$0 \longrightarrow R \xrightarrow{\delta^0} \mathfrak{X}^1(R) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} \mathfrak{X}^p(R) \xrightarrow{\delta^p} \mathfrak{X}^{p+1}(R) \longrightarrow \cdots \tag{2.2}$$

where $\delta^p\colon \mathfrak{X}^p(R) \longrightarrow \mathfrak{X}^{p+1}(R)$ is defined as $F \mapsto \delta^p(F)$ with

$$\delta^p(F)(a_1 \wedge \cdots \wedge a_{p+1}) = \sum_{i=1}^{p+1}(-1)^i\{F(a_1 \wedge \cdots \widehat{a_i} \cdots \wedge a_{p+1}), a_i\} + \sum_{1\le i<j\le p+1}(-1)^{i+j}F(\{a_i, a_j\} \wedge a_1 \wedge \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots \wedge a_{p+1}),$$

where $\widehat{a_i}$ means that the corresponding entry a_i is omitted.

Definition 2.2. *[11, 18] The complex* (2.2) *is called the* **Poisson cochain complex** *of R, and its p-th cohomology is called the p-th* **Poisson cohomology** *of R, denoted by* $\mathrm{HP}^p(R)$.

2.2. Pseudo-unimodular Poisson algebras

We first recall the definitions of contraction maps for any commutative algebra and the modular derivation for a smooth Poisson algebra with trivial canonical bundle.

Definition 2.3. *For any $\omega \in \Omega^p(R)$, the* **contraction map** *ι_ω is a graded R-linear map of degree $-p$ on the complex* $\mathrm{Hom}_R(\Omega^*(R), R)$, *which is defined as* $\iota_\omega : \mathrm{Hom}_R(\Omega^q(R), R) \to \mathrm{Hom}_R(\Omega^{q-p}(R), R)$: *when $q < p$, $\iota_\omega = 0$; when $q \ge p$ and $F \in \mathrm{Hom}_R(\Omega^q(R), R)$,*

$$(\iota_\omega F)(\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{q-p}) = F(\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{q-p} \wedge \omega).$$

By (2.1), *this induces an operation ι_ω on $\mathfrak{X}^*(R)$.*

Definition 2.4. *For any $F \in \mathfrak{X}^p(R)$, the* **contraction map** *$\iota_F : \Omega^*(R) \to \Omega^*(R)$ is a graded R-linear map of degree $-p$, which is defined as $\iota_F : \Omega^q(R) \to \Omega^{q-p}(R)$: when $q < p$, $\iota_F = 0$; when $q \ge p$ and $\omega = a_0\, \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_q \in \Omega^q(R)$,*

$$\iota_F(\omega) = \sum_{\sigma\in S_{p,q-p}} \mathrm{sgn}(\sigma)a_0F(a_{\sigma(1)} \wedge a_{\sigma(2)} \wedge \cdots \wedge a_{\sigma(p)})\, \mathrm{d}a_{\sigma(p+1)} \wedge \cdots \wedge \mathrm{d}a_{\sigma(q)},$$

where $S_{p,q-p}$ denotes the set of all $(p, q-p)$-shuffles, which are the permutations $\sigma \in S_q$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(q)$.

Definition 2.5. *Let R be a smooth Poisson algebra of dimension n with trivial canonical bundle $\Omega^n(R) = R\,\mathrm{vol}$ where* vol *is a volume form. The* **modular derivation** *of R with respect to* vol *is defined as the map $\phi_{\mathrm{vol}} : R \to R$ such that for any $a \in R$,*

$$\phi_{\mathrm{vol}}(a) = \frac{\mathscr{L}_{H_a}(\mathrm{vol})}{\mathrm{vol}},$$

where $H_a = \{a, -\} : R \to R$ is the Hamiltonian derivation associated to a and $\mathscr{L}_{H_a} = [\mathrm{d}, \iota_{H_a}]$ is the Lie derivation.

When the volume form is changed, the corresponding modular derivation is modified by a so called *log-Hamiltonian derivation* (see [8]). The *modular class* of R is defined as the class ϕ_{vol} modulo log-Hamiltonian derivations. If the modular class is trivial, i.e., ϕ_{vol} is a log-Hamiltonian derivation, then R is said to be *unimodular*.

Definition 2.6. *[21] A Poisson algebra (R, π) is said to be* **pseudo-unimodular** *if there exists a de Rham 1-cocycle $\varpi \in \Omega^1(R)$ such that $\iota_\varpi\pi$ is the modular derivation of R.*

Remark 2.7. *In this paper, we consider the pseudo-unimodular Poisson structures which are more general than the unimodular ones and share similar homological properties.*

2.3. Gerstenhaber algebras and Batalin-Vilkovisky algebras

Let $G = \bigoplus_{i\in\mathbb{Z}} G_i$ be a graded vector space, and $G(1)$ be the shift of G such that $G(1)_i = G_{i+1}$.

Definition 2.8. *[9] A* **Gerstenhaber algebra** $(G, \cdot, [-,-])$ *is a graded-commutative algebra* $(G = \bigoplus_{i\in\mathbb{Z}} G_i, \cdot)$ *endowed with a bracket of degree* 0

$$[-,-] : G(1) \times G(1) \to G(1),$$

such that $G(1)$ *is a graded Lie algebra, and for any homogeneous element* $a \in G(1)$*, the map* $[a,-]$ *is a graded-derivation, i.e., for any homogeneous elements* $b, c \in G$,

$$[a, b\cdot c] = [a,b]\cdot c + (-1)^{(|a|-1)|b|} b \cdot [a,c].$$

Example 2.9. *For any commutative algebra* R, $(\mathfrak{X}^*(R), \wedge, [-,-]_{\mathbf{SN}})$ *is a Gerstenhaber algebra (see [16]), where* $[-,-]_{\mathbf{SN}} : \mathfrak{X}^p(R) \times \mathfrak{X}^q(R) \to \mathfrak{X}^{p+q-1}(R)$ *is the* **Schouten-Nijenhuis** *bracket : for any* $P \in \mathfrak{X}^p(R)$ *and* $Q \in \mathfrak{X}^q(R)$,

$$\begin{aligned}[P,Q]_{\mathbf{SN}}(a_1\wedge a_2\wedge\cdots\wedge a_{p+q-1}) =& (-1)^{(p-1)(q-1)}\sum_{\sigma\in S_{q,p-1}} \mathrm{sgn}(\sigma) P(Q(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(q)})\wedge a_{\sigma(q+1)}\wedge\cdots\wedge a_{\sigma(p+q-1)})\\ &-\sum_{\sigma\in S_{p,q-1}} \mathrm{sgn}(\sigma) Q(P(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)})\wedge a_{\sigma(p+1)}\wedge\cdots\wedge a_{\sigma(p+q-1)}).\end{aligned}$$

Example 2.10. *([16, Proposition 4.9]) For any Poisson algebra* R*, its Poisson cohomology* $(\mathrm{HP}^*(R), \wedge, [-,-]_{\mathbf{SN}})$ *is also a Gerstenhaber algebra.*

Definition 2.11. *([17, Proposition-Definition 2.2]) Let* $(A^\bullet, \wedge_A, [-,-]_A)$ *and* $(B^\bullet, \wedge_B, [-,-]_B)$ *be two Gerstenhaber algebras. Then there is a new Gerstenhaber algebra* $(L^\bullet, \wedge, [-,-])$ *given as follows:*

1. $L^n = \bigoplus_{i+j=n} A^i \otimes B^j$ *as a* $\Bbbk$*-vector space for* $n \in \mathbb{Z}$;
2. $(a\otimes b)\wedge(a'\otimes b') = (-1)^{|a'||b|}(a\wedge_A a')\otimes(b\wedge_B b')$;
3. $[a\otimes b, a'\otimes b'] = (-1)^{(|a'|-1)|b|}[a,a']_A\otimes(b\wedge_B b') + (-1)^{|a'|(|b|-1)}(a\wedge_A a')\otimes[b,b']_B$,

where $a, a' \in A^\bullet$ *and* $b, b' \in B^\bullet$ *are homogeneous elements. We call* $(L^\bullet, \wedge, [-,-])$ *the* **tensor Gerstenhaber algebra** *of* $A^\bullet$ *and* $B^\bullet$*, and denote it by* $A^\bullet \otimes B^\bullet$.

Example 2.12. *Let* R *and* S *be two Poisson algebras, then* $\mathrm{HP}^*(R)\otimes\mathrm{HP}^*(S)$ *is a tensor Gerstenhaber algebra.*

Definition 2.13. *[1–3] Let* $(V,\cdot)$ *be a graded-commutative graded algebra. A Batalin-Vilkovisky operator* Δ *on* V *is an operator* $\Delta : V \to V$ *of degree* -1 *such that* $\Delta^2 = 0$ *and*

$$[a,b] := (-1)^{|a|}\Big(\Delta(a\cdot b) - \Delta(a)\cdot b - (-1)^{|a|} a\cdot\Delta(b)\Big) \tag{2.3}$$

is a graded-derivation in the sense that for any homogeneous elements $a, b, c \in V$,

$$[a, b\cdot c] = [a,b]\cdot c + (-1)^{(|a|-1)|b|} b\cdot[a,c].$$

In other words, the obstruction of Δ *from being a graded-derivation is a graded-derivation. The triple* $(V, \cdot, \Delta)$ *is called a* **Batalin-Vilkovisky algebra** *(BV algebra, for short).*

Remark 2.14. *Batalin-Vilkovisky algebras are special examples of Gerstenhaber algebras if one define the Lie bracket by* (2.3)*. A Gerstenhaber algebra with the bracket* $[-,-]$ *is a BV algebra (or said to be exact) if it can be equipped with an operator* Δ *of degree* -1 *such that* $\Delta^2 = 0$ *and* (2.3) *holds. In other words,* $[-,-]$ *measures the deviation of* Δ *from being a derivation.*

Example 2.15. *([16, 21, 33]) For any smooth algebra* R *with trivial canonical bundle, the triple* $(\mathfrak{X}^*(R), \wedge, \Delta)$ *is a BV algebra, where the BV operator* Δ *can be described by using the dual basis of the Kähler differential module. When* R *is a unimodular smooth Poisson algebra, its Poisson cohomology* $\mathrm{HP}^*(R)$ *admits a BV algebra structure induced from the one on* $\mathfrak{X}^*(R)$.

Let us recall the tensor product of two BV algebras defined in [23, Proposition in Section 5.8.1].

Proposition-Definition 2.16. *[23, Proposition in Section 5.8.1] Let $(A^\bullet, \wedge_A, \Delta_A)$ and $(B^\bullet, \wedge_B, \Delta_B)$ be two BV algebras. Then there is a new BV algebra $(L^\bullet, \wedge, \Delta_\otimes)$ given as follows:*

1. $L^n = \bigoplus_{i+j=n} A^i \otimes B^j$ *as a* $\Bbbk$*-vector space for* $n \in \mathbb{Z}$;
2. $(a \otimes b) \wedge (a' \otimes b') = (-1)^{|a'||b|}(a \wedge_A a') \otimes (b \wedge_B b')$;
3. $\Delta_\otimes(a \otimes b) = \Delta_A(a) \otimes b + (-1)^{|a|} a \otimes \Delta_B(b)$,

where $a, a' \in A^\bullet$ and $b, b' \in B^\bullet$ are homogeneous elements. We call $(L^\bullet, \wedge, \Delta_\otimes)$ the **tensor BV algebra** *of $A^\bullet$ and $B^\bullet$, and denote it by $A^\bullet \otimes B^\bullet$.*

Remark 2.17. *In the above definition, $(L^\bullet, \wedge, [-,-])$ is the tensor Gerstenhaber algebra of the two Gerstenhaber algebras $(A^\bullet, \wedge_A, [-,-]_A)$ and $(B^\bullet, \wedge_B, [-,-]_B)$, where $[-,-]$, $[-,-]_A$ and $[-,-]_B$ are defined by* (2.3) *with corresponding $\wedge, \wedge_A, \wedge_B$ and $\Delta_\otimes, \Delta_A, \Delta_B$.*

Example 2.18. *Let R and S be two unimodular Poisson algebras. There is a tensor BV algebra* $\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(S)$.

3. The cohomology of tensor Poisson algebras

In this section, we focus on the tensor products of Poisson algebras and their cohomologies. Throughout this section, all the Poisson algebras involved are smooth affine algebras with trivial canonical bundle.

3.1. Tensor Poisson algebras

Let $(R, \{-,-\}_R)$ and $(S, \{-,-\}_S)$ be Poisson algebras. Then the tenor product $R \otimes S$ admits a Poisson structure $\{-,-\}$ given by

$$\{r \otimes s, r' \otimes s'\} = \{r, r'\}_R \otimes ss' + rr' \otimes \{s, s'\}_S, \forall r, r' \in R, s, s' \in S.$$

The Poisson algebra $(R \otimes S, \{-,-\})$ is called the tensor Poisson algebra of R and S.

Suppose that R and S are smooth Poisson algebras with trivial canonical bundles. Then the tensor Poisson algebra $R \otimes S$ is still a smooth algebra with trivial canonical bundle, and the modular derivation ϕ of $R \otimes S$ is closely related to the modular derivations of R and S:

$$\phi(r \otimes s) = \phi_R(r) \otimes s + r \otimes \phi_S(s), \forall r \otimes s \in R \otimes S, \tag{3.1}$$

where ϕ_R and ϕ_S are the modular derivations of R and S, respectively. See [30, Theorem 4.2].

Remark 3.1. *For the tensor algebra $R \otimes S$, its Kähler differential module $\Omega^1(R \otimes S) \cong (\Omega^1(R) \otimes S) \oplus (R \otimes \Omega^1(S))$, and the classical de Rham differential $\mathrm{d} : R \otimes S \to \Omega^1(R \otimes S)$ is given by $\mathrm{d}(r \otimes s) = \mathrm{d}\, r \otimes s + r \otimes \mathrm{d}\, s$.*

Proposition 3.2. *If R and S are pseudo-unimodular Poisson algebras, then so is the tensor Poisson algebra $R \otimes S$.*

Proof. By the definition of pseudo-unimodular Poisson algebras, there exist de Rham 1-cocycles $\varpi \in \Omega^1(R)$ and $\omega \in \Omega^1(S)$ such that the modular derivations $\phi_R = \iota_\varpi \pi_R$, $\phi_S = \iota_\omega \pi_S$.

Consider the 1-form $\varpi \otimes 1 + 1 \otimes \omega \in \Omega^1(R \otimes S)$, it's easy to check that it is a de Rham 1-cocycle in $\Omega^1(R \otimes S)$. We claim that the modular derivation $\phi = \iota_{(\varpi\otimes 1 + 1\otimes\omega)}\pi$, which implies that $R \otimes S$ is also pseudo-unimodular.

For any $r \otimes s \in R \otimes S$,

$$\begin{aligned}
\phi(r \otimes s) &= \phi_R(r) \otimes s + r \otimes \phi_S(s) \\
&= \iota_\varpi \pi_R(r) \otimes s + r \otimes \iota_\omega \pi_S(s) \\
&= \pi_R(\mathrm{d}\, r \wedge \varpi) \otimes s + r \otimes \pi_S(\mathrm{d}\, s \wedge \omega) \\
&= \pi(\mathrm{d}(r \otimes s) \wedge \varpi \otimes 1) + \pi(\mathrm{d}(r \otimes s) \wedge 1 \otimes \omega) \\
&= \iota_{(\varpi\otimes 1 + 1\otimes\omega)}\pi(r \otimes s).
\end{aligned}$$

Remark 3.3. *For a smooth Poisson algebra R with trivial canonical bundle, its Poisson cohomology admits a BV algebra structure induced from* $\mathfrak{X}^*(R)$*, if and only if R is pseudo-unimodular ([21, Theorem 5.9 and Corollary 5.12]).*

Corollary 3.4. *If R and S are pseudo-unimodular Poisson algebras, then the Poisson cohomologies* $\mathrm{HP}^*(R), \mathrm{HP}^*(S)$ *and* $\mathrm{HP}^*(R \otimes S)$ *are all BV algebras.*

Corollary 3.5. *For pseudo-unimodular Poisson algebras R and S, their Poisson cohomology rings carry a BV algebra structure and induce the structure of a BV algebra* $\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(S)$ *as the tensor product of two BV algebras (See 2.16).*

Now we fix some notations to describe the BV operators. Let R be a smooth Poisson algebra of dimension n, $\{(\mathrm{d}x_i), (\mathrm{d}x_i)^*\}_{i=1}^r$ be a dual basis for $\Omega^1(R)$, and $\eta \in \Omega^n(R)$ be a volume form. Let $U = \{(I_1, I_2, \cdots, I_n) \mid I_1, \cdots, I_n$ are integers and $1 \le I_1 < I_2 < \cdots < I_n \le r\}$. For any $I = (I_1, I_2, \cdots, I_n) \in U$, to simplify the notations, let $\mathrm{d}x_I$ denote $\mathrm{d}x_{I_1} \wedge \mathrm{d}x_{I_2} \wedge \cdots \wedge \mathrm{d}x_{I_n}$ and $\mathrm{d}x_I^*$ denote $(\mathrm{d}x_{I_1})^* \wedge (\mathrm{d}x_{I_2})^* \wedge \cdots \wedge (\mathrm{d}x_{I_n})^*$. Let

$$a_I = (\mathrm{d}x_I^*)(\eta) \quad \text{and} \quad b_I = \eta^*(\mathrm{d}x_I).$$

Then $(\mathfrak{X}^*(R), \wedge, \Delta)$ is a BV algebra with the BV operator Δ given by

$$\Delta(P)(a_1 \wedge a_2 \wedge \cdots \wedge a_{p-1}) = (-1)^p \sum_{1 \le l \le r} (\mathrm{d}x_l)^* \Big(P(a_1 \wedge a_2 \wedge \cdots \wedge a_{p-1} \wedge x_l) \Big) + (-1)^p \sum_{I \in U} P(a_1 \wedge a_2 \wedge \cdots \wedge a_{p-1} \wedge a_I) b_I,$$

for each $P \in \mathfrak{X}^p(R)$ (see [21, Theorem 4.15] for details).

Similarly, let S be a smooth Poisson algebra of dimension m, $\{(\mathrm{d}y_i), (\mathrm{d}y_i)^*\}_{i=1}^s$ be a dual basis for $\Omega^1(S)$, and $\xi \in \Omega^m(S)$ be a volume form. Let $V = \{(J_1, J_2, \cdots, J_m) \mid J_1, \cdots, J_m$ are integers and $1 \le J_1 < J_2 < \cdots < J_m \le s\}$. For any $J = (J_1, J_2, \cdots, J_m) \in V$, let $\mathrm{d}y_J$ denote $\mathrm{d}y_{J_1} \wedge \mathrm{d}y_{J_2} \wedge \cdots \wedge \mathrm{d}y_{J_m}$ and $\mathrm{d}y_J^*$ denote $(\mathrm{d}y_{J_1})^* \wedge (\mathrm{d}y_{J_2})^* \wedge \cdots \wedge (\mathrm{d}y_{J_m})^*$. Let

$$c_J = (\mathrm{d}y_J^*)(\xi) \quad \text{and} \quad d_J = \xi^*(\mathrm{d}y_J).$$

Then the BV operator on $\mathfrak{X}^*(S)$ given by

$$\Delta(Q)(b_1 \wedge b_2 \wedge \cdots \wedge b_{q-1}) = (-1)^q \sum_{1 \le l \le s} (\mathrm{d}y_l)^* \Big(Q(b_1 \wedge b_2 \wedge \cdots \wedge b_{p-1} \wedge y_l) \Big) + (-1)^q \sum_{J \in V} P(b_1 \wedge b_2 \wedge \cdots \wedge b_{p-1} \wedge c_J) d_J,$$

for each $Q \in \mathfrak{X}^q(S)$.

In this setting, $R \otimes S$ is a smooth Poisson algebra of dimension $n + m$ with $\{\mathrm{d}x_i \otimes 1, 1 \otimes \mathrm{d}y_j; (\mathrm{d}x_i)^* \otimes 1, 1 \otimes (\mathrm{d}y_j)^*\}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$ a dual basis for $\Omega^1(R \otimes S)$ and $\eta \otimes \xi \in \Omega^{m+n}(R \otimes S)$ a volume form. For any $I \in U, J \in V$,

$$(\mathrm{d}x_I^* \otimes \mathrm{d}y_J^*)(\eta \otimes \xi) = a_I \otimes c_J \quad \text{and} \quad (\eta \otimes \xi)^*(\mathrm{d}x_I \otimes \mathrm{d}y_J) = b_I \otimes d_J.$$

Then the BV operator on $\mathfrak{X}^*(R \otimes S)$ is given by

$$\begin{aligned}\Delta(T)(c_1 \wedge c_2 \wedge \cdots \wedge c_{t-1}) =& (-1)^t \sum_{1 \le i \le r} ((\mathrm{d}x_i)^* \otimes 1) \Big(T(c_1 \wedge c_2 \wedge \cdots \wedge c_{p-1} \wedge (\mathrm{d}x_i \otimes 1)) \Big) \\ &+ (-1)^t \sum_{1 \le j \le s} (1 \otimes (\mathrm{d}y_j)^*) \Big(T(c_1 \wedge c_2 \wedge \cdots \wedge c_{p-1} \wedge (1 \otimes \mathrm{d}y_j)) \Big) \\ &+ (-1)^t \sum_{I \in U, J \in V} T(c_1 \wedge c_2 \wedge \cdots \wedge c_{p-1} \wedge (a_I \otimes c_J))(b_I \otimes d_J),\end{aligned}$$

for each $T \in \mathfrak{X}^t(R \otimes S)$.

3.2. BV structures on Poisson cohomology

For two Poisson algebras R and S, it is proved that there is an isomorphism of Gerstenhaber algebras [35]

$$\mathrm{HP}^*(R)\otimes\mathrm{HP}^*(S)\cong\mathrm{HP}^*(R\otimes S),$$

which is induced from $\mathfrak{X}^*(R)\otimes\mathfrak{X}^*(S)\cong\mathfrak{X}^*(R\otimes S)$. The isomorphism

$$\Phi:\mathfrak{X}^*(R)\otimes\mathfrak{X}^*(S)\to\mathfrak{X}^*(R\otimes S)$$

is constructed as $\bigoplus_{p,q}\Phi_{p,q}$, where $\Phi_{p,q}:\mathfrak{X}^p(R)\otimes\mathfrak{X}^q(S)\to\mathfrak{X}^{p+q}(R\otimes S)$ is defined by

$$\begin{aligned}\Phi_{p,q}(f\otimes g):&(a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q}\otimes b_{p+q})\mapsto\\ &\sum_{\sigma\in S_{p,q}}\mathrm{sgn}(\sigma)a_{\sigma(p+1)}\cdots a_{\sigma(p+q)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)})\otimes b_{\sigma(1)}\cdots b_{\sigma(p)}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)}),\end{aligned}$$

for any $f\in\mathfrak{X}^p(R)$ and $g\in\mathfrak{X}^q(S)$.

Theorem 3.6. *Let R and S be unimodular Poisson algebras. There is an isomorphism of BV algebras*

$$\mathrm{HP}^*(R)\otimes\mathrm{HP}^*(S)\cong\mathrm{HP}^*(R\otimes S).$$

Proof. The left hand side of the above isomorphism is the tensor product of two BV algebras, while the right hand side is the cohomology of the tensor Poisson algebra $R\otimes S$. In the unimodular Poisson algebras case, the BV operators of Poisson cohomologies $\mathrm{HP}^*(R)$, $\mathrm{HP}^*(S)$ and $\mathrm{HP}^*(R\otimes S)$ are induced by the BV operators on $\mathfrak{X}^*(R)$, $\mathfrak{X}^*(S)$ and $\mathfrak{X}^*(R\otimes S)$, respectively. Since $\Phi:\mathfrak{X}^*(R)\otimes\mathfrak{X}^*(S)\to\mathfrak{X}^*(R\otimes S)$ induces the isomorphism as Gerstenhaber algebras ([35]), we only need to prove that Φ preserves the BV operator:

$$\begin{array}{ccc}\mathfrak{X}^*(R)\otimes\mathfrak{X}^*(S) & \xrightarrow{\Phi} & \mathfrak{X}^*(R\otimes S)\\ \downarrow{\scriptstyle\Delta_\otimes} & & \downarrow{\scriptstyle\Delta}\\ \mathfrak{X}^*(R)\otimes\mathfrak{X}^*(S) & \xrightarrow{\Phi} & \mathfrak{X}^*(R\otimes S),\end{array}$$

i.e. $\Delta\Phi(f\otimes g)=\Phi\Delta_\otimes(f\otimes g)$ for any $f\in\mathfrak{X}^p(R)$and $g\in\mathfrak{X}^q(S)$.

By the definition of $\Delta_\otimes$, $\Delta_\otimes(f\otimes g)=\Delta f\otimes g+(-1)^pf\otimes\Delta g$. Then

$$\Phi\Delta_\otimes(f\otimes g)=\Phi_{p-1,q}(\Delta f\otimes g)+(-1)^p\Phi_{p,q-1}(f\otimes\Delta g).$$

For any $a_1,\cdots,a_{p+q-1}\in R$, and $b_1,\cdots,b_{p+q-1}\in S$,

$$\begin{aligned}&\Phi\Delta_\otimes(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1}))\\ =&\sum_{\sigma\in S_{p-1,q}}\mathrm{sgn}(\sigma)a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}\Delta f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)})\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\ &+(-1)^p\sum_{\sigma\in S_{p,q-1}}\mathrm{sgn}(\sigma)a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)})\otimes b_{\sigma(1)}\cdots b_{\sigma(p)}\Delta g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\ =&(-1)^p\sum_{\sigma\in S_{p-1,q}}\sum_{1\le i\le r}\mathrm{sgn}(\sigma)a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}(\mathrm{d}x_i)^*\Big(f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\Big)\\ &\quad\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\ &+(-1)^p\sum_{\sigma\in S_{p-1,q}}\sum_{I}\mathrm{sgn}(\sigma)a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_I)b_I\\ &\quad\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\end{aligned}$$

$$+ (-1)^{p+q} \sum_{\sigma\in S_{p,q-1}} \sum_{1\le j\le s} \mathrm{sgn}(\sigma) a_{\sigma(p+1)} \cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p)})$$
$$\otimes b_{\sigma(1)} \cdots b_{\sigma(p)} (\mathrm{d}\, y_j)^* \Big(g(b_{\sigma(p+1)} \wedge \cdots \wedge b_{\sigma(p+q-1)} \wedge y_j) \Big)$$
$$+ (-1)^{p+q} \sum_{\sigma\in S_{p,q-1}} \sum_{J} \mathrm{sgn}(\sigma) a_{\sigma(p+1)} \cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p)})$$
$$\otimes b_{\sigma(1)} \cdots b_{\sigma(p)} g(b_{\sigma(p+1)} \wedge \cdots \wedge b_{\sigma(p+q-1)} \wedge c_J) d_J,$$

which are denoted by $B1, B2, B3$ and $B4$, respectively. While

$$\begin{aligned}
&\Delta\Phi(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1}))\\
=&(-1)^{p+q}\sum_{1\le i\le r}((\mathrm{d}x_i)^*\otimes 1)\Big(\Phi(f\otimes g)((a_1\otimes b_1)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})\wedge(x_i\otimes 1))\Big)\\
&+(-1)^{p+q}\sum_{1\le j\le s}(1\otimes(\mathrm{d}y_j)^*)\Big(\Phi(f\otimes g)((a_1\otimes b_1)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})\wedge(1\otimes y_j))\Big)\\
&+(-1)^{p+q}\sum_{I,J}\Phi(f\otimes g)((a_1\otimes b_1)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})\wedge(a_I\otimes c_J))(b_I\otimes d_J)
\end{aligned}$$

Denote the three parts of the sum by $A1, A2$ and $A3$, respectively. Then

$$\begin{aligned}
A1=&(-1)^{p+q}\sum_{1\le i\le r}((\mathrm{d}x_i)^*\otimes 1)\Big(\sum_{\substack{\sigma\in S_{p,q}\\ \sigma(p)=p+q}}\mathrm{sgn}(\sigma)a_{\sigma(p+1)}\cdots a_{\sigma(p+q)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\Big)\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}\cdot 1\cdot g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)})\\
=&(-1)^{p+q}\sum_{1\le i\le r}((\mathrm{d}x_i)^*\otimes 1)\Big(\sum_{\sigma\in S_{p-1,q}}(-1)^q\,\mathrm{sgn}(\sigma)a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\Big)\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\
=&(-1)^{p}\sum_{\sigma\in S_{p-1,q}}\sum_{1\le i\le r}\mathrm{sgn}(\sigma)(\mathrm{d}x_i)^*\Big(a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\Big)\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\
=&(-1)^{p}\sum_{\sigma\in S_{p-1,q}}\sum_{1\le i\le r}\mathrm{sgn}(\sigma)a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}(\mathrm{d}x_i)^*\Big(f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\Big)\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\
&+(-1)^{p}\sum_{\sigma\in S_{p-1,q}}\sum_{1\le i\le r}\mathrm{sgn}(\sigma)(\mathrm{d}x_i)^*\Big(a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}\Big)f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge x_i)\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}).
\end{aligned}$$

Note that the first part is exactly $B1$. Denote the second part by $A11$, and

$$\begin{aligned}
A11=&(-1)^{p}\sum_{\sigma\in S_{p-1,q}}\mathrm{sgn}(\sigma)f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{\sigma(p)}\cdots a_{\sigma(p+q-1)})\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)})\\
=&(-1)^{p}\sum_{\sigma\in S_{p-1,q}}\sum_{p\le k\le p+q-1}\mathrm{sgn}(\sigma)f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{\sigma(k)})a_{\sigma(p)}\cdots\widehat{a_{\sigma(k)}}\cdots a_{\sigma(p+q-1)}\\
&\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}).
\end{aligned}$$

Similarly, one can check that $A2 = A22 + B3$, where

$$\begin{aligned} A22 =&(-1)^{p+q} \sum_{\sigma\in S_{p,q-1}} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}) \\ &\otimes g\Big(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge b_{\sigma(1)}\cdots b_{\sigma(p)}\Big) \\ =&(-1)^{p+q} \sum_{\sigma\in S_{p,q-1}}\sum_{1\le l\le p} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}) \\ &\otimes g\Big(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge b_{\sigma(l)}\Big) b_{\sigma(1)}\cdots\widehat{b_{\sigma(l)}}\cdots b_{\sigma(p)}. \end{aligned}$$

Considering the 1-1 correspondence between $\{(\sigma,k)|\sigma\in S_{p-1,q}, p\le k\le p+q-1\}$ and $\{(\sigma',l)|\sigma'\in S_{p,q-1}, 1\le l\le p\}$, we can get $A11 + A22 = 0$. Thus $A1 + A2 = B1 + B3$.

$$\begin{aligned} A3 =&(-1)^{p+q}\sum_{I,J}\Phi(f\otimes g)((a_1\otimes b_1)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})\wedge(a_I\otimes c_J))(b_I\otimes d_J) \\ =&(-1)^{p+q}\sum_{I,J}\Big(\sum_{\substack{\sigma\in S_{p,q}\\ \sigma(p)=p+q}} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_I) \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)} c_J g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)})\Big)(b_I\otimes d_J) \\ &+(-1)^{p+q}\sum_{I,J}\Big(\sum_{\substack{\sigma\in S_{p,q}\\ \sigma(p+q)=p+q}} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} a_I f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}) \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p)} g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge c_J)\Big)(b_I\otimes d_J) \\ =&(-1)^{p}\sum_{I,J}\sum_{\sigma\in S_{p-1,q}} \operatorname{sgn}(\sigma) a_{\sigma(p)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_I) b_I \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)} c_J g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}) d_J \\ &+(-1)^{p+q}\sum_{I,J}\sum_{\sigma\in S_{p,q-1}} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} a_I f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}) b_I \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p)} g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge c_J) d_J \end{aligned}$$

Note that $\sum_I a_I b_I = 1, \sum_J c_J d_J = 1$. So

$$\begin{aligned} A3 =&(-1)^{p}\sum_{I}\sum_{\sigma\in S_{p-1,q}} \operatorname{sgn}(\sigma) a_{\sigma(p)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_I) b_I \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}) \\ &+(-1)^{p+q}\sum_{J}\sum_{\sigma\in S_{p,q-1}} \operatorname{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}) \\ &\otimes b_{\sigma(1)}\cdots b_{\sigma(p)} g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge c_J) d_J \\ =&B2 + B4. \end{aligned}$$

Hence $A1 + A2 + A3 = B1 + B2 + B3 + B4$. The proof is finished.

3.3. Pseudo-unimodular Poisson algebras case

Recall that if (R,π) is a pseudo-unimodular Poisson algebra, i.e., there exists a de Rham 1-cocycle $\varpi\in\Omega^1(R)$ such that its modular derivation $\phi_{\mathrm{vol}} = \iota_\varpi\pi$, then $(\mathrm{HP}^*(R),\wedge,\Delta_t)$ is also a BV algebra induced from $(\mathfrak{X}^*(R), \wedge, \Delta_t)$, where

$$\Delta_t = \Delta - \Delta',\ \Delta'(P) = (-1)^p \iota_{\varpi} P \tag{3.2}$$

(see [21, Theorem 5.9 and Remark 5.10]). Now consider the tensor Poisson algebra of two pseudo-unimodular Poisson algebras, and we have the following theorem.

Theorem 3.7. *Let R and S be pseudo-unimodular Poisson algebras. There is an isomorphism of BV algebras*

$$\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(S) \cong \mathrm{HP}^*(R \otimes S).$$

Proof. Again using the fact that $\Phi : \mathfrak{X}^*(R) \otimes \mathfrak{X}^*(S) \to \mathfrak{X}^*(R \otimes S)$ induce the isomorphism as Gerstenhaber algebras ([35]), we should prove that the isomorphism Φ preserves the twisted BV operator, that is,

$$\Delta_t \Phi(f \otimes g) = \Phi \Delta_{t\otimes}(f \otimes g) = \Phi_{p-1,q}(\Delta_t f \otimes g) + (-1)^p \Phi_{p,q-1}(f \otimes \Delta_t g)$$

for any $f \in \mathfrak{X}^p(R)$ and $g \in \mathfrak{X}^q(S)$.

Note that the BV operators in pseudo-unimodular Poisson algebras case are twisted as in (3.2). Suppose $\varpi \in \Omega^1(R)$ and $\omega \in \Omega^1(S)$ are de Rham 1-cocycles such that the modular derivations $\phi_R = \iota_{\varpi}\pi_R$, $\phi_S = \iota_{\omega}\pi_S$. Then

$$\Delta_t f = \Delta f - (-1)^p \iota_{\varpi} f, \Delta_t g = \Delta g - (-1)^q \iota_{\omega} g.$$

From the proof of Proposition 3.2,

$$\Delta_t \Phi(f \otimes g) = (\Delta - (-1)^{p+q} \iota_{\varpi\otimes 1 + 1\otimes\omega})(\Phi(f \otimes g)).$$

In Theorem 3.6, we have proved that the isomorphism Φ preserves the operator Δ. Thus we only need to prove that

$$(-1)^{p+q} \iota_{\varpi\otimes 1+1\otimes\omega}(\Phi(f \otimes g)) = (-1)^p \Phi_{p-1,q}(\iota_{\varpi} f \otimes g) + (-1)^{p+q} \Phi_{p,q-1}(f \otimes \iota_{\omega} g),$$

i.e.

$$\iota_{\varpi\otimes 1}(\Phi(f \otimes g)) + \iota_{1\otimes\omega}(\Phi(f \otimes g)) = (-1)^q \Phi_{p-1,q}(\iota_{\varpi} f \otimes g) + \Phi_{p,q-1}(f \otimes \iota_{\omega} g).$$

In fact, $\iota_{\varpi\otimes 1}(\Phi(f \otimes g))$ is the map sending $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) \wedge \cdots \wedge (a_{p+q-1} \otimes b_{p+q-1})$ to

$$\begin{aligned}
&\Phi(f \otimes g)((a_1 \otimes b_1) \wedge (a_2 \otimes b_2) \wedge \cdots \wedge (a_{p+q-1} \otimes b_{p+q-1}) \wedge (\varpi \otimes 1)) \\
=& \sum_{\substack{\sigma \in S_{p,q} \\ \sigma(p)=p+q}} \mathrm{sgn}(\sigma) a_{\sigma(p+1)} \cdots a_{\sigma(p+q)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p-1)} \wedge \varpi) \otimes b_{\sigma(1)} \cdots b_{\sigma(p-1)} \cdot 1 \cdot g(b_{\sigma(p+1)} \wedge \cdots \wedge b_{\sigma(p+q)}) \\
=& \sum_{\sigma \in S_{p-1,q}} (-1)^q \,\mathrm{sgn}(\sigma) a_{\sigma(p)} \cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p-1)} \wedge \varpi) \otimes b_{\sigma(1)} \cdots b_{\sigma(p-1)} g(b_{\sigma(p)} \wedge \cdots \wedge b_{\sigma(p+q-1)}) \\
=&(-1)^q \sum_{\sigma \in S_{p-1,q}} \mathrm{sgn}(\sigma) a_{\sigma(p)} \cdots a_{\sigma(p+q-1)} \iota_{\varpi} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p-1)}) \otimes b_{\sigma(1)} \cdots b_{\sigma(p-1)} g(b_{\sigma(p)} \wedge \cdots \wedge b_{\sigma(p+q-1)}) \\
=&(-1)^q \Phi_{p-1,q}(\iota_{\varpi} f \otimes g)\big((a_1 \otimes b_1) \wedge (a_2 \otimes b_2) \wedge \cdots \wedge (a_{p+q-1} \otimes b_{p+q-1})\big).
\end{aligned}$$

Hence $\iota_{\varpi\otimes 1}(\Phi(f \otimes g)) = (-1)^q \Phi_{p-1,q}(\iota_{\varpi} f \otimes g)$. Similarly, $\iota_{1\otimes\omega}(\Phi(f \otimes g)) = \Phi_{p,q-1}(f \otimes \iota_{\omega} g)$. Thus we finish the proof.

4. Tensor product of Frobenius Poisson algebras

In fact, tensor Poisson algebras can also be defined for Frobenius Poisson algebras. In this section, we consider the tensor products of Frobenius Poisson algebras and conclude the similar results in the previous section. Let us first recall the modular derivations for Frobenius Poisson algebras, which are different from those of smooth Poisson algebras.

Definition 4.1. *[31] Let R be a Frobenius Poisson algebra with the non-degenerated bilinear form $\langle -,-\rangle$. Its* **modular derivation** *is defined as the map $D: R \to R$ such that, for any $a \in R$ and $x \in R$,*

$$\langle D(a), x\rangle = \langle 1, \{a, x\}\rangle.$$

For any unimodular Frobenius Poisson algebra R, $(\mathrm{HP}^*(R), \wedge, \Delta)$ is a BV-algebra ([36, Theorem 4.10]). Note that the BV operator here is different from the one in smooth algebras case.

However, pseudo-unimodular Poisson structures can also be defined for Frobenius algebras similarly.

Definition 4.2. *([20, Definition 10]) Let R be a Frobenius Poisson algebra with Poisson structure π. Then R is said to be* **pseudo-unimodular** *if there exists a de Rham 1-cocycle $\varpi \in \Omega^1(R)$ such that $\iota_\varpi \pi$ is a modular derivation of R.*

Let R be a pseudo-unimodular Frobenius Poisson algebra with poisson structure π. Then $(\mathrm{HP}^*(R), \wedge, \Delta_t)$ is a BV-algebra, where

$$\Delta_t = \Delta - \Delta',\ \Delta'(P) = (-1)^{p-1}\iota_\varpi(P), \tag{4.1}$$

$\varpi \in \Omega^1(R)$ is the de Rham 1-cocycle such that $\iota_\varpi \pi$ is a modular derivation of R (see[20, Theorem 2]). Furthermore, for a Frobenius Poisson algebra R, its Poisson cohomology admits a BV-operator induced from $\mathfrak{X}^*(R)$, iff it is pseudo-unimodular ([20, Corollary 2]).

Suppose that R and S are Frobenius Poisson algebras. Then the tensor Poisson algebra $R \otimes S$ is also a Frobenius Poisson algebra, and equation(3.1)

$$\phi(r \otimes s) = \phi_R(r) \otimes s + r \otimes \phi_S(s), \forall r \otimes s \in R \otimes S,$$

still holds (see [30, Theorem 4.4]). Thus Proposition 3.2 is also valid in Frobenius algebras case. That is

Proposition 4.3. *If R and S are pseudo-unimodular Frobenius Poisson algebras, then so is the tensor Poisson algebra $R \otimes S$.*

Corollary 4.4. *If R and S are pseudo-unimodular Frobenius Poisson algebras, then the Poisson cohomologies $\mathrm{HP}^*(R), \mathrm{HP}^*(S), \mathrm{HP}^*(R \otimes S)$ are all BV algebras, and $\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(S)$ carries the structure of a BV algebra as the tensor product of two BV algebras.*

In the following, we'll prove that $\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(S) \cong \mathrm{HP}^*(R \otimes S)$ as BV algebras in the corollary above.

Now we fix some notations. In the following, R is a Frobenius Poisson algebra with the non-degenerated bilinear form $\langle -,-\rangle_R$. Then $(\mathfrak{X}^*(R), \wedge, \Delta)$ is a Batalin-Vilkovisky algebra ([36, Theorem 4.5]), where for $P \in \mathfrak{X}^p(R)$, $\Delta(P) \in \mathfrak{X}^{p-1}(R)$ is given by

$$\langle \Delta(P)(a_1 \wedge a_2 \wedge \cdots \wedge a_{p-1}), a_p\rangle_R = (-1)^{p-1}\langle P(a_1 \wedge a_2 \wedge \cdots \wedge a_p), 1\rangle_R.$$

Suppose S is also a Frobenius Poisson algebra with the non-degenerated bilinear form $\langle -,-\rangle_S$. Then the BV operator on $\mathfrak{X}^*(S)$ is given by

$$\langle \Delta(Q)(b_1 \wedge b_2 \wedge \cdots \wedge b_{q-1}), b_p\rangle_S = (-1)^{q-1}\langle Q(b_1 \wedge b_2 \wedge \cdots \wedge b_q), 1\rangle_S.$$

for any $Q \in \mathfrak{X}^q(S)$.

Consider the tensor algebra $R \otimes S$. It's still a Frobenius Poisson algebra with the non-degenerated bilinear form $\langle -,-\rangle$:

$$\langle r \otimes s, r' \otimes s'\rangle := \langle r, r'\rangle_R \langle s, s'\rangle_S, \forall r, r' \in R, s, s' \in S.$$

The BV operator on $\mathfrak{X}^*(R \otimes S)$ is given by

$$\langle \Delta(T)(c_1 \wedge c_2 \wedge \cdots \wedge c_{t-1}), c_t\rangle = (-1)^{t-1}\langle T(c_1 \wedge c_2 \wedge \cdots \wedge c_t), 1 \otimes 1\rangle.$$

for any $T \in \mathfrak{X}^t(R \otimes S)$.

Theorem 4.5. *Let R and S be pseudo-unimodular Frobenius Poisson algebras. There is an isomorphism of BV algebras*

$$\mathrm{HP}^*(R)\otimes \mathrm{HP}^*(S)\cong \mathrm{HP}^*(R\otimes S).$$

Proof. Similar to the smooth algebras case, we should prove the isomorphism Φ preserves the BV operator in the Frobenius algebras case,
i.e.

$$\Delta\Phi(f\otimes g)=\Phi\Delta_{\otimes}(f\otimes g)=\Phi_{p-1,q}(\Delta f\otimes g)+(-1)^p\Phi_{p,q-1}(f\otimes\Delta g), \tag{4.2}$$

and

$$\Delta_t\Phi(f\otimes g)=\Phi\Delta_{t\otimes}(f\otimes g)=\Phi_{p-1,q}(\Delta_t f\otimes g)+(-1)^p\Phi_{p,q-1}(f\otimes\Delta_t g),$$

for any $f\in\mathfrak{X}^p(R)$ and $g\in\mathfrak{X}^q(S)$. Here Δ_t is the twisted BV operator for pseudo-unimodular Frobenius Poisson algebra (see (4.1)). Note that the twisted part Δ' is similar to the smooth algebras case (3.2), and we've proved that Φ preserves Δ' in Theorem 3.7. Hence, we only need to prove (4.2).

By the definition of BV operators Δ, it suffices to show that

$$\begin{aligned}&\langle\Phi\Delta_{\otimes}(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})),a_{p+q}\otimes b_{p+q}\rangle\\&=(-1)^{(p+q-1)}\langle\Phi(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q}\otimes b_{p+q})),1\otimes 1\rangle.\end{aligned}$$

In fact,

$$\begin{aligned}&(-1)^{(p+q-1)}\langle\Phi(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q}\otimes b_{p+q})),1\otimes 1\rangle\\
=&(-1)^{(p+q-1)}\langle\sum_{\sigma\in S_{p,q}}\mathrm{sgn}(\sigma)a_{\sigma(p+1)}\cdots a_{\sigma(p+q)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)})\otimes b_{\sigma(1)}\cdots b_{\sigma(p)}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)}),1\otimes 1\rangle\\
=&(-1)^{(p+q-1)}\sum_{\sigma\in S_{p,q}}\mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}),1\rangle_R\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p)}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)}),1\rangle_S\\
=&(-1)^{(p+q-1)}\sum_{\substack{\sigma\in S_{p,q}\\ \sigma(p)=p+q}}\mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{p+q}),1\rangle_R\\
&\quad\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)}b_{p+q}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q)}),1\rangle_S\\
&+(-1)^{(p+q-1)}\sum_{\substack{\sigma\in S_{p,q}\\ \sigma(p+q)=p+q}}\mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)}a_{p+q}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}),1\rangle_R\\
&\quad\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p)}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge b_{p+q}),1\rangle_S\\
=&(-1)^{(p-1)}\sum_{\sigma\in S_{p-1,q}}\mathrm{sgn}(\sigma)\langle a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{p+q}),1\rangle_R\\
&\quad\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)}b_{p+q}g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}),1\rangle_S\\
&+(-1)^{(p+q-1)}\sum_{\sigma\in S_{p,q-1}}\mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)}a_{p+q}f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}),1\rangle_R\\
&\quad\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p)}g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge b_{p+q}),1\rangle_S\\
=&B1+B2,\end{aligned}$$

where $B1$ and $B2$ denote the two parts of the sum respectively.

While

$$\begin{aligned}&\langle\Phi\Delta_{\otimes}(f\otimes g)((a_1\otimes b_1)\wedge(a_2\otimes b_2)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})),a_{p+q}\otimes b_{p+q}\rangle\\
=&\langle\Big(\Phi_{p-1,q}(\Delta f\otimes g)+(-1)^p\Phi_{p,q-1}(f\otimes\Delta g)\Big)\Big((a_1\otimes b_1)\wedge\cdots\wedge(a_{p+q-1}\otimes b_{p+q-1})\Big),a_{p+q}\otimes b_{p+q}\rangle\end{aligned}$$

$$
\begin{aligned}
=&\langle \sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma) a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}\Delta f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)})\\
&\quad \otimes b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), a_{p+q}\otimes b_{p+q}\rangle\\
&+\langle(-1)^p \sum_{\sigma\in S_{p,q-1}} \mathrm{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)})\\
&\quad \otimes b_{\sigma(1)}\cdots b_{\sigma(p)}\Delta g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), a_{p+q}\otimes b_{p+q}\rangle\\
=&\langle \sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma) a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}\Delta f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}), a_{p+q}\rangle_R\\
&\quad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S\\
&+\langle(-1)^p \sum_{\sigma\in S_{p,q-1}} \mathrm{sgn}(\sigma) a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}), a_{p+q}\rangle_R\\
&\quad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p)}\Delta g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S\\
=&\sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma)\langle \Delta f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}), a_{\sigma(p)}\cdots a_{\sigma(p+q-1)} a_{p+q}\rangle_R\\
&\quad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S\\
&+(-1)^p \sum_{\sigma\in S_{p,q-1}} \mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}), a_{p+q}\rangle_R\\
&\quad \cdot\langle \Delta g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{\sigma(1)}\cdots b_{\sigma(p)} b_{p+q}\rangle_S\\
=&(-1)^{p-1} \sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma)\langle f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{\sigma(p)}\cdots a_{\sigma(p+q-1)} a_{p+q}), 1\rangle_R\\
&\quad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S\\
&+(-1)^{p+q-1} \sum_{\sigma\in S_{p,q-1}} \mathrm{sgn}(\sigma)\langle a_{\sigma(p+1)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p)}), a_{p+q}\rangle_R\\
&\quad \cdot\langle g(b_{\sigma(p+1)}\wedge\cdots\wedge b_{\sigma(p+q-1)}\wedge b_{\sigma(1)}\cdots b_{\sigma(p)} b_{p+q}), 1\rangle_S\\
=&A1 + A2,
\end{aligned}
$$

where $A1$ and $A2$ denote the two parts of the sum respectively.

Since f is a p-derivation,

$$
\begin{aligned}
A1 = (-1)^{p-1} &\sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma)\langle a_{\sigma(p)}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{p+q}), 1\rangle_R\\
&\qquad\qquad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} b_{p+q} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), 1\rangle_S\\
+(-1)^{p-1} &\sum_{\sigma\in S_{p-1,q}} \mathrm{sgn}(\sigma)\langle f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{\sigma(p)}\cdots a_{\sigma(p+q-1)}), a_{p+q}\rangle_R\\
&\qquad\qquad \cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S.
\end{aligned}
$$

Note that the first part of the sum is exactly $B1$. Denote the second part by $A11$, and

$$
\begin{aligned}
A11 =(-1)^{p-1} &\sum_{\sigma\in S_{p-1,q}} \sum_{p\le k\le p+q-1} \mathrm{sgn}(\sigma)\langle a_{\sigma(p)}\cdots \widehat{a_{\sigma(k)}}\cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p-1)}\wedge a_{\sigma(k)}), a_{p+q}\rangle_R\\
&\cdot\langle b_{\sigma(1)}\cdots b_{\sigma(p-1)} g(b_{\sigma(p)}\wedge\cdots\wedge b_{\sigma(p+q-1)}), b_{p+q}\rangle_S.
\end{aligned}
$$

Similarly, one can check that $A2 = B2 + A22$, where

$$\begin{aligned} A22 &= (-1)^{p+q-1} \sum_{\sigma \in S_{p,q-1}} \mathrm{sgn}(\sigma) \langle a_{\sigma(p+1)} \cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p)}), a_{p+q} \rangle_R \\ &\qquad \cdot \langle g\Big(b_{\sigma(p+1)} \wedge \cdots \wedge b_{\sigma(p+q-1)} \wedge b_{\sigma(1)} \cdots b_{\sigma(p)}\Big), b_{p+q} \rangle_S \\ &= (-1)^{p+q-1} \sum_{\sigma \in S_{p,q-1}} \sum_{1 \le l \le p} \mathrm{sgn}(\sigma) \langle a_{\sigma(p+1)} \cdots a_{\sigma(p+q-1)} f(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p)}), a_{p+q} \rangle_R \\ &\qquad \cdot \langle b_{\sigma(1)} \cdots \widehat{b_{\sigma(l)}} \cdots b_{\sigma(p)} g\Big(b_{\sigma(p+1)} \wedge \cdots \wedge b_{\sigma(p+q-1)} \wedge b_{\sigma(l)}\Big), b_{p+q} \rangle_S. \end{aligned}$$

Considering the 1-1 correspondence between $\{(\sigma, k) | \sigma \in S_{p-1,q}, p \le k \le p+q-1\}$ and $\{(\sigma', l) | \sigma' \in S_{p,q-1}, 1 \le l \le p\}$, we can get $A11 + A22 = 0$. Thus $A1 + A2 = B1 + B2$. The proof is finished.

Acknowledgment

We would like to thank the anonymous referees for their valuable comments and suggestions that led to the improvement of the manuscript. J. Luo is supported by the NSFC (Grant No. 11901396). S.-Q. Wang is supported by the NSFC (Grant No. 11771085).

References

[1] I. A. Batalin, G. S. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, Phys. Rev. D, **28** (1983), 2567–2582.
[2] I. A. Batalin, G. S. Vilkovisky, *Closure of the gauge algebra, generalized Lie equations and Feynman rules*, Nuclear Phys. B, **234** (1984), 106–124.
[3] I. A. Batalin, G. S. Vilkovisky, *Existence theorem for gauge algebra*, J. Math. Phys, **26** (1985), 172–184.
[4] R. Berger, A. Pichereau, *Calabi-Yau algebras viewed as deformations of Poisson algebras*, Algebra Represent. Theory **17** (2014), 735–773.
[5] X. Chen, Y. Chen, F. Eshmatov and S. Yang, *Poisson cohomology, Koszul duality, and Batalin-Vilkovisky algebras*, J. Noncommut. Geom. **15** (2021), 889–918.
[6] S.-S. Chen, T.-T. Chen and J. Luo, *Batalin-Vilkovisky structures on the cohomology of polynomial Poisson algebras*, preprint.
[7] V. P. Cherkashin, *Tensor products of Poisson algebras*, Math. Notes **54** (1993), 968–974.
[8] V. A. Dolgushev, *The Van den Bergh duality and the modular symmetry of a Poisson variety*, Selecta Math. **14** (2009), 199–228.
[9] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267-288.
[10] V. Ginzburg, *Lectures on Noncommutative Geometry*, arXiv: math. AG/0506603.
[11] J. Huebschmann, *Poisson cohomology and quantization*, J. Reine Angew. Math. **408** (1990), 57–113.
[12] J. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, J. Reine Angew. Math. **510** (1999), 103–159.
[13] N. Kowalzig, U. Krähmer, *Batalin-Vilkovisky structures on Ext and Tor*, J. Reine Angew. Math. **697** (2014), 159-219.
[14] T. Lambre, G. Zhou and A. Zimmermann, *The Hochschild cohomology ring of a Frobenius algebra with semisimple Nakayama automorphism is a Batalin-Vilkovisky algebra*, J. Algebra **446** (2016), 103-131.
[15] S. Launois, L. Richard, *Twisted Poincaré duality for some quadratic Poisson algebras*, Lett. Math. Phys. **79** (2007), 161–174.
[16] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke, *Poisson structures*, Springer, Berlin, 2013.
[17] J. Le and G. Zhou, *On the Hochschild cohomology ring of tensor products of algebras*, J. Pure Appl. Algebra. **218** (2014), 1463–1477.
[18] A. Lichnerowicz, *Les varieties de Poisson et leurs algebres de Lie associees (French)*, J. Differential Geom. **12** (1977), 253–300.
[19] J. Luo, S.-Q. Wang and Q.-S. Wu, *Twisted Poincaré duality between Poisson homology and Poisson cohomology*, J. Algebra **442** (2015), 484–505.
[20] J. Luo, S.-Q. Wang and Q.-S. Wu, *Frobenius Poisson algebras*, Front. Math. China **14** (2019), 395–420.
[21] J. Luo, S.-Q. Wang and Q.-S. Wu, *Poincaré duality for smooth Poisson algebras and BV structure on Poisson cohomology*, J. Algebra **649** (2024), 169–211.
[22] J. Lü, X. Wang and G. Zhuang, *A note on the duality between Poisson homology and cohomology*, Comm. Algebra. **48** (2020), 4170–4175.
[23] Y. I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, 47, American Mathematical Society, Providence, RI, 1999.
[24] N. Marconnet, *Homologies of cubic Artin-Schelter regular algebras*, J. Algebra **278** (2004), 638–665.
[25] P. Mnev, *Quantum Field Theory: Batalin–Vilkovisky Formalism and Its Applications*, University Lecture Series, Vol. 72 (2019), Amer. Math. Soc.
[26] S. R. Tagne Pelap, *On the Hochschild homology of elliptic Sklyanin algebras*, Lett. Math. Phys. **87** (2009), 267–281.
[27] S. R. Tagne Pelap, *Poisson (co)homology of polynomial Poisson algebras in dimension four: Sklyanin's case*, J. Algebra **322** (2009), 1151–1169.

[28] T. Tradler, *The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products*, Ann. Inst. Fourier **58** (2008), 2351-2379.
[29] M. Van den Bergh, *Noncommutative homology of some three-dimensional quantum spaces*, *K*-Theory **8** (1994), 213–230.
[30] S.-Q. Wang, *Modular Derivations for Extensions of Poisson Algebras*, Front. Math. China, **12** (2017), 209–218.
[31] S.-Q. Wang, Q.-S. Wu , G.-D. Zhou, C. Zhu, *Batalin-Vilkovisky structure over Frobenius Poisson cohomology*, preprint.
[32] A. Weinstein, *Lecture on symplectic manifolds*, CBMS Conference series in Math. 29, 1977.
[33] P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm. Math. Phys. **200** (1999), 545–560.
[34] C. Zhu, *Twisted Poincaré duality for Poisson homology and cohomology of affine Poisson algebras*, Proc. Amer. Math. Soc. **143** (2015), 1957–1967.
[35] C. Zhu, *Cohomology ring of the tensor product of Poisson algebras*, J. Korean Math. Soc. **57** (2020), 113–129.
[36] C. Zhu, Oystaeyen F. Van, Y H. Zhang, *On (co)homology of Frobenius Poisson algebras*. J. K-Theory, **14** (2014), 371–386.